Crystalline cohomology and applications

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DRAFT

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Foreword

These are lectures notes for a graduate course I taught in Easter Term 2013 at the University of Cambridge. The subject was crystalline cohomology. A concrete approach is taken, working directly with de Rham complexes. These notes are a work-in-progress.

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CHAPTER 1

Introduction

1. The algebraic de Rham complex

Let R be a ring and A an algebra over R. Recall the definition of the module of Kähler differentials of A over R. It is an A-module $\Omega^1_{A/R}$ equipped with an R-linear map $d : A \to \Omega^1_{A/R}$ satisfying

$$d(a_1 a_2) = a_1 da_2 + a_2 da_1 \quad (a_1, a_2 \in A).$$

We say that d is an R-linear *derivation.* Moreover, d is the universal such derivation: given any other R-linear map $D : A \to M$ into an A-module M satisfying $D(a_1a_2) = a_1D(a_2) + a_2D(a_1)$ for $a_1, a_2 \in A$, there is a unique A-linear map $\theta : \Omega^1_{A/R} \to M$ such that $D = \theta d$.

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[d, "d"'] & M \\ \Omega^1_{A/R} \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

We define the module of differential n-forms to be $\Omega^n_{A/R} = \bigwedge^n \Omega^1_{A/R}$. We extend d to a map

$$d : \Omega^n_{A/R} \to \Omega^{n+1}_{A/R}.$$

by setting $d(da) = 0$ for every $a \in A$ and using the formula

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p(\omega_1 \wedge d\omega_2)$$

when $\omega_1 \in \Omega^p_{A/R}$ and $\omega_2 \in \Omega^{n-p}_{A/R}$.

A simple calculation shows that the composition $d^2 : \Omega^n_{A/R} \to \Omega^{n+2}_{A/R}$ is zero. Thus we obtain a complex, the *algebraic de Rham complex of A/R:*

$$\Omega^\bullet_{A/R} : A \to \Omega^1_{A/R} \to \Omega^2_{A/R} \to \cdots \to \Omega^n_{A/R} \to \cdots$$

Next we globalise this construction. Let X be a scheme over a ring R. There is a unique quasi-coherent sheaf $\Omega^1_{X/R}$ on X such that for every open affine $U = \operatorname{Spec} A \subset X$ there is a functorial identification

$$\Gamma(U, \Omega^1_{X/R}) = \Omega^1_{A/R}.$$

As before, we have a complex *of R-modules* (d is only R-linear!) on X:

$$\Omega^\bullet_{X/R} : \mathscr{O}_X \to \Omega^1_{X/R} \to \Omega^2_{X/R} \to \cdots \to \Omega^n_{X/R} \to \cdots$$

By definition the *algebraic de Rham cohomology of X/R* is the hypercohomology of this complex:

$$H^*_{dR}(X/R) = \mathbb{H}^*(X, \Omega^\bullet_{X/R}).$$

Recall that this means that one has to choose an injective resolution $\Omega^\bullet_{X/R} \to I^\bullet$ in the category of abelian sheaves on X and then set $\mathbb{H}^*(X, \Omega^\bullet_{X/R}) = H^*(\Gamma(X, I^\bullet))$. One then has to prove that this is well-defined and functorial.

EXERCISE 1.1. If $X = \operatorname{Spec} A$ is an affine scheme over R, prove that its algebraic de Rham cohomology can be computed directly in terms of the de Rham complex: $H^*_{dR}(X/R) = H^*(\Omega^\bullet_{A/R})$.

2. de Rham cohomology over $\mathbb{C}$

Let X_0 be a scheme over $\mathbb{C}$. If X_0 is smooth over $\mathbb{C}$, A. Grothendieck proved [5] that the algebraic de Rham cohomology of X_0 over $\mathbb{C}$ coincides with its topological singular cohomology:

$$H^*_{dR}(X_0/\mathbb{C}) = H^*_{sing}(X_0(\mathbb{C})^{an}, \mathbb{C}).$$

However, this is false if $X_0/\mathbb{C}$ is singular. See [1, Example 4.4] for an example.

Hence $\mathbb{H}^*(X_0, \Omega^\bullet_{X_0/\mathbb{C}})$ *is not the correct object to consider in this case.* How to solve this problem? Deligne had the idea of embedding X_0 inside a smooth X. Consider a diagram

$$\begin{tikzcd} X_0 \arrow[r, hook, "\text{closed}"] \arrow[d] & X \arrow[dl, "\text{smooth}"] \\ \operatorname{Spec}\mathbb{C} & \end{tikzcd}$$

Let $\widehat{\Omega}^\bullet_{X/\mathbb{C}}$ be the *completion* of $\Omega^\bullet_{X/\mathbb{C}}$ *along* X_0. This means the following. Let $\mathscr{I}$ be the ideal defining X_0 as a closed subscheme of X. Then $\mathscr{I}^{n+1}$ defines a closed subscheme $X_n \subset X$. Since $\mathscr{I}/\mathscr{I}^{n+1}$ is a nilpotent ideal in $\mathscr{O}_{X_{n+1}}$ we see that the underlying topological space of any of the spaces in the sequence $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ coincide and thus $\Omega^\bullet_{X_n/\mathbb{C}}$ is simply a complex of sheaves of $\mathbb{C}$-vector spaces on X_0. Define

$$\widehat{\Omega}^\bullet_{X/\mathbb{C}} = \varprojlim_{n\to\infty} \Omega^\bullet_{X_n/\mathbb{C}}.$$

This is a complex of sheaves of $\mathbb{C}$-vector spaces on X_0.

EXERCISE 1.2. Let B be a $\mathbb{C}$-algebra and let $I \subset B$ be an ideal. Let $\widehat{B}$ be the I-adic completion of B. Let $\rho : \Omega^1_{B/\mathbb{C}} \to \Omega^1_{\widehat{B}/\mathbb{C}}$ be the canonical homomorphism.

i) Show that we have $\Omega^1_{\widehat{B}/\mathbb{C}} = \rho(\Omega^1_{B/\mathbb{C}}) + I\Omega^1_{\widehat{B}/\mathbb{C}}$. Deduce that for any $n \geq 1$, ρ induces a short exact sequence

$$0 \to K_n \to \Omega^1_{B/\mathbb{C}}/I^n\Omega^1_{B/\mathbb{C}} \to \Omega^1_{\widehat{B}/\mathbb{C}}/I^n\Omega^1_{\widehat{B}/\mathbb{C}} \to 0$$

for some B-module K_n. Furthermore, the natural map $K_{n+1} \to K_n$ is surjective for every $n \geq 1$.

ii) Suppose that B is of finitely generated over $\mathbb{C}$. Show that

$$\varprojlim_{n\to\infty} \Omega^1_{\widehat{B}/\mathbb{C}}/I^n\Omega^1_{\widehat{B}/\mathbb{C}} \simeq \Omega^1_{B/\mathbb{C}} \otimes_B \widehat{B} \simeq \widehat{\Omega}^1_{B/\mathbb{C}}.$$

(the last isomorphism is well-known from basic commutative algebra)

Deligne's observation is the following.

THEOREM 1.3. *The complex $\widehat{\Omega}^\bullet_{X/\mathbb{C}}$ is independent of X up to canonical quasi-isomorphism.*

The key point is the following special case of the above theorem. Suppose that we have a diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[dr] & X \arrow[r, hook] \arrow[d] & X' \arrow[dl] \\ & \operatorname{Spec}\mathbb{C} & \end{tikzcd}$$

where the horizontal maps are closed embeddings and both X and X' are smooth over $\mathbb{C}$.

THEOREM 1.4. *Suppose that $X \to X'$ is a closed embedding of smooth schemes over $\mathbb{C}$. The induced map $\widehat{\Omega}^\bullet_{X'/\mathbb{C}} \to \widehat{\Omega}^\bullet_{X/\mathbb{C}}$ is a quasi-isomorphism of sheaves of $\mathbb{C}$-vector spaces on X_0.*

The idea is that after we complete, X' looks like an affine space over X, and then the statement follows from an explicit calculation (the *Poincaré Lemma* below). Since it is a local statement, we can assume that X_0, X, and X' are all affine schemes, say $X_0 = \operatorname{Spec} A_0$, $X = \operatorname{Spec} B$, $X' = \operatorname{Spec} B'$.

$$\begin{tikzcd} A_0 & B \arrow[l, two heads] & B' \arrow[l, two heads] \\ & \mathbb{C} \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Let $J = \ker(B \to A_0)$ and $J' = \ker(B' \to A_0)$. Recall that a closed embedding between smooth schemes is locally defined by a regular sequence. This means that we can assume that $B = B'/(x_1, \dots, x_N)$ where x_i is not a zero-divisor in $B'/(x_1, \dots, x_{i-1})$ for $i = 1, \dots, N$. By induction, we can assume $N = 1$, so that $B = B'/x$.

The smoothness of B over $\mathbb{C}$ will imply the following fact.

LEMMA 1.5. *Let $\widehat{B}$ be the J-adic completion of B and let $\widehat{B}'$ be the J'-adic completion of B'. There is a (non-canonical) isomorphism $\widehat{B}[[T]] \simeq \widehat{B}'$ sending T to x and making the following diagram commute:*

$$\begin{tikzcd} \widehat{B}' \arrow[r, two heads] & \widehat{B} \\ \widehat{B}[[T]] \arrow[u, "\simeq"] \arrow[ur, "T \mapsto 0"'] & \end{tikzcd}$$

PROOF. We use the smoothness of B over $\mathbb{C}$ to define a compatible system of ring homomorphisms $f_i : B \to B'/x^i$ by filling in the diagrams below inductively, starting with $f_1 = \mathrm{id} : \operatorname{Spec} B'/x \to \operatorname{Spec} B$.

$$\begin{tikzcd} \operatorname{Spec} B'/x^n \arrow[r, "f_n"] \arrow[d, hook] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} B'/x^{n+1} \arrow[r] \arrow[ur, dashed, "f_{n+1}"] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

We deduce a map $f : B \to \widehat{B}'_x$ that lifts the identity $B \to B'/x$. In particular, f is injective. Since $x \to 0$ in $\widehat{B}'_x$, we can extend f to a map $g : B[[T]] \to \widehat{B}'_x$, sending T to x. This map is injective because x is not a zero-divisor in B'[1]. And it is surjective because $B = B'/x$[2].

Finally, since $J' \supset xB'$ and $J'/xJ' = J$, we see that

$$\widehat{B}' \simeq \varprojlim_{n\to\infty} B'/J'^n \simeq \varprojlim_{n\to\infty} B[[T]]/J^n \simeq \widehat{B}[[T]]. \quad \square$$

We obtain the following diagram

$$\begin{tikzcd}
\widehat{\Omega}^\bullet_{\widehat{B}'/\mathbb{C}} \arrow[r, two heads] & \widehat{\Omega}^\bullet_{\widehat{B}/\mathbb{C}} \\
\widehat{\Omega}^\bullet_{\widehat{B}[[T]]/\mathbb{C}} \arrow[u, "\simeq"] \arrow[ur, two heads, "T\mapsto 0"'] &
\end{tikzcd}$$

(where the completion $\widehat{\Omega}^\bullet_{B'/\mathbb{C}}$ refers to the J'-adic topology, $\widehat{\Omega}^\bullet_{\widehat{B}[[T]]/\mathbb{C}}$ to the (J,T)-adic topology and $\widehat{\Omega}^\bullet_{B/\mathbb{C}}$ to the J-adic topology). By Exercise 1.2, it suffices to prove the following lemma.

Lemma 1.6 (Poincaré Lemma). *Let B be a $\mathbb{C}$-algebra. The map $\Omega^\bullet_{B[T]/\mathbb{C}} \to \Omega^\bullet_{B/\mathbb{C}}$ induced by sending T to 0 is a quasi-isomorphism.*

Proof. Let $\pi : B[T] \to B$ be the map sending T to 0 and let $j : B \to B[T]$ be the natural inclusion. Then the natural short exact sequence

$$0 \to \Omega^1_{B/\mathbb{C}} \otimes_B B[T] \to \Omega^1_{B[T]/\mathbb{C}} \to \Omega^1_{B[T]/B} \to 0$$

is split by j and therefore π and j induce an isomorphism

$$\Omega^\bullet_{B[T]/\mathbb{C}} \simeq \Omega^\bullet_{B/\mathbb{C}} \otimes_B \Omega^\bullet_{B[T]/B}.$$

Hence it is enough to prove that the natural map $B \to \Omega^\bullet_{B[T]/B}$ is a quasi-isomorphism, i.e., that the sequence

$$0 \to B \to B[T] \to B[T]\,dT \to 0$$

is exact. Exactness at the first and second term is clear. Exactness for the last term means that every element in $B[T]\,dT$ is a boundary. But $T^{i+1}/(i+1)$ maps to $T^i\,dT$ so we are done. $\square$

Remark 1.7. It is precisely the last part of the proof, where we "integrated" $T^i\,dT$ that fails for bases other than $\mathbb{C}$ (for example $\mathbb{Z}$ or $\mathbb{Z}_p$). This "problem" gives rise to crystalline cohomology, as we will explain in the next section.

[1]If $\sum f(b_i)x^i = g(\sum b_i T_i) = 0$, then by reducing modulo x^n for different n's and using the x is not a zero divisor in B' we see that all the $f(b_i)$, and hence all the b_i, must be zero.

[2]It is enough to check that $g : B[[T]] \to B'/x^n$ is surjective for every n. This follows by an easy induction using the short exact sequence $0 \to x^i/x^{i+1} \to B'/x^{i+1} \to B'/x^i \to 0$.

3. The idea of crystalline cohomology

What happens in characteristic $p > 0$? Suppose that X_0 is a *smooth* scheme over $\mathbb{F}_p$. One can always consider the usual de Rham cohomology of $X_0/\mathbb{F}_p$, $H^*_{dR}(X_0/\mathbb{F}_p)$. Unfortunately, this will only give numerical information *modulo* p. Because of this, we would like a theory with values in characteristic zero. The following idea arises: consider X_0 as a *singular* scheme over $\mathbb{Z}_p$ and embed it inside a scheme X, smooth over $\mathbb{Z}_p$, as before.

$$\begin{tikzcd} X_0 \arrow[r, hook, "closed"] \arrow[d] & X \arrow[d, "smooth"] \\ \operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}_p \end{tikzcd}$$

Question. Is $\widehat{\Omega}^\bullet_{X/\mathbb{Z}_p}$ independent of X?

Answer. *No!* Because the Poincaré Lemma does not hold. For example, when $X_0 = \operatorname{Spec} \mathbb{F}_p$, $X = \operatorname{Spec} \mathbb{Z}_p$, $X' = \operatorname{Spec} \mathbb{Z}_p[T]$, we are led to consider the following de Rham complex:

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p[T] \to \mathbb{Z}_p[T]\, dT \to 0.$$

This complex is not exact! For example, $T^{p-1}dT$ is not a boundary because $T^p/p! \notin \mathbb{Z}_p[T]$.

How to fix this? Replace $\mathbb{Z}_p[T]$ by $\mathbb{Z}_p[T]\,\langle T^n/n!\rangle_{n\geq 1} \subset \mathbb{Q}_p[T]$. One can easily check that

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p[T]\left\langle \frac{T^n}{n!}\right\rangle_{n\geq 1} \to \mathbb{Z}_p[T]\left\langle \frac{T^n}{n!}\right\rangle_{n\geq 1} dT \to 0$$

is again exact. *We have restored the validity of the Poincaré Lemma by adding some divided powers to a suitable ideal.*

4. Bibliographic notes

Deligne's definition of the de Rham cohomology of a singular variety over $\mathbb{C}$ is developed by R. Hartshorne in [6]. The same result was reproven in spectacular fashion by Grothendieck in [5] and he suggested how to modify the definitions to deal with the case of positive characteristic. This work was carried out in Berthelot's thesis [2] (see also [3]).

CHAPTER 2

Divided power structures

In this chapter we introduce divided power structures and prove the existence of the divided power analog of polynomial rings (divided power polynomial algebras).

1. Definition

DEFINITION 2.1. Let R be a ring, $I \subset R$ an ideal. A *divided power structure* on (R, I) is a sequence of maps $\gamma_n : I \to R$, $n \geq 0$ such that

(DP1) $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_n(x) \in I$ for every $x \in I$, $n \geq 1$.
(DP2) $\gamma_n(x)\gamma_m(x) = \binom{n+m}{n}\gamma_{n+m}(x)$ for every $x \in I$, $n, m \geq 0$.
(DP3) $\gamma_n(ax) = a^n\gamma_n(x)$ for every $a \in R$, $x \in I$, $n \geq 0$.
(DP4) $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(x)$ for every $x, y \in I$, $n \geq 0$.
(DP5) $\gamma_n(\gamma_m(x)) = (nm)!/(m!)^n n!\gamma_{nm}(x)$ for every $x \in I$, $n \geq 0$, $m > 0$.

REMARK 2.2. The coefficient $(nm)!/(m!)^n n!$ is an integer if $m > 0$, as can be seen by using the inductive formula

$$\frac{(nm)!}{(m!)^n n!} = \frac{((n-1)m)!}{(m!)^{n-1}(n-1)!}\binom{nm-1}{m-1}.$$

REMARK 2.3. If $\gamma = (\gamma_n)_n$ is a divided power structure on (R, I) and $x \in I$, then

$$x\gamma_{n-1}(x) \overset{\text{(DP1)}}{=} \gamma_1(x)\gamma_{n-1}(x) \overset{\text{(DP2)}}{=} n\gamma_n(x).$$

An easy induction shows that $x^n = n!\gamma_n(x)$.

REMARK 2.4. If R is torsion free, it follows from Remark 2.3 that there is at most one divided power structure on any ideal I. Furthermore, I admits a divided power structure if and only if for every $x \in I$ and $n \geq 1$, $x^n \in n!I$. This condition can even be checked on generators: suppose the condition holds for $x, y \in I$, i.e., $x^n \in n!I$ and $y^n \in n!I$. Then for any $r \in R$ we get $(rx+y)^n = \sum_{h=0}^{n} \binom{n}{h} r^h x^h y^{n-h} \in n!I$ if $n \geq 1$ so the condition also holds for $Rx + y$.

REMARK 2.5. Let R be a $\mathbb{Q}$-algebra. Then by Remark 2.3, any ideal $I \subset R$ has a unique divided power structure given by $\gamma_n(x) = x^n/n!$.

REMARK 2.6. Let γ and γ' be divided power structures on $I = \langle x_\alpha \rangle_\alpha$. Then $\gamma = \gamma'$ if and only if $\gamma(x_\alpha) = \gamma'(x_\alpha)$ for every α. Suppose that $\gamma(x) = \gamma'(x)$ and $\gamma(y) = \gamma'(y)$. Then for $r \in R$, we get $\gamma_n(rx + y) = \sum_{i+j=n} r^i\gamma_i(x)\gamma_j(y) = \sum_{i+j=n} r^i\gamma'_i(x)\gamma'_j(y) = \gamma'_n(rx + y)$. Thus the subset of elements of I where γ and γ' agree is an ideal and contains the x_α, so it must coincide with I.

2. Quotients of divided power rings

The following lemma is elementary but will come in handy many times.

DEFINITION 2.7. Let (R, I, γ) be a divided power ring. An ideal $J \subset R$ is a *divided power ideal* if $\gamma_n(I \cap J) \subset I \cap J$ for every $n \geq 1$.

REMARK 2.8. In the notation of the definition, suppose that $I \cap J = \langle x_\alpha \rangle_\alpha$. Then J is a divided power ideal if and only if $\gamma_n(x_\alpha) \in I \cap J$ for every $n \geq 1$. Indeed, if $\gamma_n(x), \gamma_n(y) \in I \cap J$ for every $n \geq 1$ then $\gamma_n(rx + y) \in I \cap J$ for every $n \geq 1$, $r \in R$ by using (DP3) and (DP4) in the usual manner.

LEMMA 2.9. *Let (R, I, γ) be a divided power ring, $J \subset R$ an ideal. Then γ extends to $(R/J, I/I \cap J)$ if and only if J is a divided power ideal. Any such extension is unique.*

PROOF. Suppose first that there is a divided power structure $\overline{\gamma}$ extending γ. Then $\gamma_n(x) \equiv \overline{\gamma}_n(x) \equiv 0 \pmod{I \cap J}$ so $\gamma_n(x) \in I \cap J$ for every $x \in I \cap J$, $n \geq 1$.

To prove the converse define $\overline{\gamma}_n(\overline{x}) = \overline{\gamma_n(x)}$ where the bar means reduction mod J. To check that it is well-defined, suppose that $z \in I \cap J$. Then $\gamma_n(x + z) = \sum_{i+j=n} \gamma_i(x)\gamma_j(z) \equiv \gamma_n(x) \pmod{I \cap J}$. Axioms (DP1-5) for $\overline{\gamma}$ are obvious. $\square$

3. Divided power polynomial algebras

Let R be a ring. Define an R-module $R\langle X_1, \ldots, X_N \rangle$ by

$$R\langle X_1, \ldots, X_N \rangle = \bigoplus_{n_1, \ldots, n_N \geq 0} R X_1^{[n_1]} \cdots X_N^{[n_N]}.$$

Define a multiplication by extending linearly the rule

$$X_1^{[n_1]} \cdots X_N^{[n_N]} \cdot X_1^{[m_1]} \cdots X_N^{[m_N]} = \binom{n_1 + m_1}{n_1} \cdots \binom{n_N + m_N}{n_N} X_1^{[n_1+m_1]} \cdots X_N^{[n_N+m_N]}.$$

This defines a ring and there is a natural ring map $R \to R\langle X_1, \ldots, X_N \rangle$ so that it is an R-algebra. Note that $1 = X_1^{[0]} \cdots X_N^{[0]}$. Let

$$R\langle X_1, \ldots, X_N \rangle_+ = \bigoplus_{\text{some } n_i > 0} R X_1^{[n_1]} \cdots X_N^{[n_N]}.$$

It is the kernel of the natural R-algebra map $R\langle X_1, \ldots, X_N \rangle \to R$ sending every $X_1^{[n_1]} \cdots X_N^{[n_N]}$ with some $n_i > 0$ to zero.

REMARK 2.10. Suppose $\varphi : R\langle X_1, \ldots, X_N \rangle \to S$ is a ring homomorphism. Then it defines elements $x_i^{[n]} = \varphi(X_i^{[n]}) \in S$, $i = 1, \ldots, N$, such that $x_i^{[n]} x_i^{[m]} = \binom{n+m}{n} x_i^{[n+m]}$. Conversely, given such a sequence of elements in S there is a unique ring homomorphism $\varphi : R\langle X_1, \ldots, X_N \rangle \to S$ such that $\varphi(X_i^{[n]}) = x_i^{[n]}$.

EXERCISE 2.11. Show that $\mathbb{Z}\langle X \rangle$ is not finitely generated as an algebra over $\mathbb{Z}$.

THEOREM 2.12. *Let R be a ring. There is a unique divided power structure γ on $R\langle X_1, \ldots, X_N \rangle_+$ such that $\gamma_n(X_i^{[1]}) = X_i^{[n]}$. The divided power ring $R\langle X_1, \ldots, X_N \rangle$ has the following universal property. There is a natural bijection between maps*

$(R\langle X_1,\dots,X_N\rangle, R\langle X_1,\dots,X_N\rangle_+, \delta) \to (S,J,\delta)$ *and ring homomorphisms* $R \to S$ *together with a choice of* N *elements* $x_1,\dots,x_N \in J$.

Proof. Suppose first that R is torsion-free. Then $R\langle X_1,\dots,X_N\rangle$ is torsion-free. By Remark 2.8, it is enough to prove that $(X_1^{[m_1]}\cdots X_N^{[m_N]})^n \in n! R\langle X_1,\dots,X_N\rangle_+$ for every n. We compute $(X_1^{[m_1]}\cdots X_N^{[m_N]})^n = \prod_{i=1}^N (nm_i)!/(m_i!)^n X_1^{[nm_1]}\cdots X_N^{[nm_N]}$ and the coefficient is divisible by $n!$ if at least one of the $m_i > 0$ (Remark 2.2) so we are done in this case. In the general case, write $R = F/K$ for some torsion-free ring F (e.g. a polynomial ring over $\mathbb{Z}$). Then $R\langle X_1,\dots,X_N\rangle$ is the quotient of $F\langle X_1,\dots,X_N\rangle$ by the ideal $\bigoplus_{m_1,\dots,m_N\geq 0} K X_1^{[m_1]}\cdots X_N^{[m_N]}$. Since this ideal is clearly preserved by the divided power structure on $F\langle X_1,\dots,X_N\rangle$, we are done by Lemma 2.9.

Lastly, we verify the universal property. Let (S,J,δ) be a divided power ring. Given a homomorphism $R \to S$ and elements $x_1,\dots,x_N \in J$, define a map $R\langle X_1,\dots,X_N\rangle \to S$ by sending $X_1^{[n_1]}\cdots X_N^{[n_N]}$ to $\delta_{n_1}(x_1)\cdots\delta_{n_N}(x_N)$. It is easy to check that this is a divided power homomorphism. □

Remark 2.13. The same construction can be carried out in the case of an arbitrary set (not necessarily finite) of variables. In fact, if Λ is any set and R is a ring, we define

$$R\langle X_\lambda : \lambda \in \Lambda\rangle = \varinjlim_{U\subset\Lambda \text{ finite}} R\langle X_u : u \in U\rangle.$$

Then $R\langle X_\lambda : \lambda\in\Lambda\rangle$ satisfies the corresponding universal property: for any homomorphism $R \to S$ and elements $x_s \in J$, $\lambda \in \Lambda$, there is a unique divided power homomorphism $R\langle X_\lambda : \lambda\in\Lambda\rangle \to S$ sending X_λ to x_λ

4. Two useful lemmas

Lemma 2.14. *Let R be a ring, $I \subset R$ and ideal. Let $(x_\alpha)_\alpha$ be a generating set for I. Suppose that $\gamma = (\gamma_n)$ is a collection of functions $\gamma_n : I \to R$ satisfying (DP1,3,4) and (DP2) and (DP5) only for $x \in \{x_\alpha\}_\alpha$. Then γ satisfies (DP1-5) (and so is a divided power structure on I).*

Proof. Suppose that $x,y \in I$ satisfy (DP2) and let $r \in R$. We need to prove that $rx+y$ satisfies (DP2). The divided powers of x and y define a ring homomorphism $\varphi : \mathbb{Z}[T]\langle X,Y\rangle \to R$ such that $\varphi(T) = r$, $\varphi(X^{[n]}) = \gamma_n(x)$, $\varphi(Y^{[m]}) = \gamma_m(y)$. It is easy to see[1] that $\varphi(\gamma_n(TX+Y)) = \gamma_n(rx+y)$. Hence it is enough to check (DP2) for $TX+Y$ in $\mathbb{Z}[T]\langle X,Y\rangle$, where it is obvious because $\overline{\gamma}$ is a divided power structure.

Next suppose that $x,y \in I$ satisfy (DP5). Let $r \in R$. We need to prove that $rx+y$ satisfies (DP5). We will use the same trick as before. The divided powers of x and y define a ring homomorphism $\varphi : \mathbb{Z}[T]\langle X,Y\rangle \to A$ such that $\varphi(T) = r$,

[1] We compute: $\gamma_n(TX+Y) = \sum_{i+j=n} T^i\gamma_i(X)\gamma_j(Y)$. Applying φ and running the computation backwards shows the desired identity.

$\varphi(X^{[n]}) = \gamma_n(x)$, $\varphi(Y^{[m]}) = \gamma_m(y)$. It is easy to see[2] that $\varphi(\gamma_n(\gamma_m(TX+Y))) = \gamma_n(\gamma_m(rx+y))$ and we already observed that $\varphi(\gamma_{nm}(TX+Y)) = \gamma_{nm}(rx+y)$. Thus it is enough to check the identity in the ring $\mathbb{Z}[T]\langle X,Y\rangle$ where it is obvious. $\square$

LEMMA 2.15. *Let R be a ring, $I, J \subset R$ ideals. Let γ be a divided power structure on (R, I) and let δ be a divided power structure on (A, J). Then $\gamma = \delta$ on IJ. And if $\delta = \gamma$ on $I \cap J$, then there is a unique divided power structure on $I + J$ that restricts to γ on I and δ on J.*

PROOF. If $x \in I$ and $y \in J$, then $\gamma_n(xy) = x^n\gamma_n(y) = n!\delta_n(x)\gamma_n(y) = y^n\delta_n(x) = \delta_n(xy)$ so $\gamma = \delta$ by Remark 2.6. This proves the first assertion.

Next, suppose that $\gamma = \delta$ on $I \cap J$. Define $\epsilon_n : I + J \to R$ as follows: if $x \in I$, $y \in J$, $\epsilon_n(x+y) = \sum_{i+j=n} \gamma_i(x)\delta_j(y)$. Clearly this is the unique possible definition if it is well-defined. To check that it is well-defined, suppose that $x + y = x' + y'$ for some $x, x' \in I$, $y, y' \in J$. Let $w = x - x' = y - y' \in I \cap J$. Then

$$\sum_{i+j=n} \gamma_i(x)\delta_j(y) = \sum_{i+j=n} \gamma_i(x'+w)\delta_j(y) = \sum_{i+j=n}\sum_{k+l=i} \gamma_k(x')\gamma_l(w)\delta_j(y) = \sum_{k+l+j=n} \gamma_k(x')\delta_l(w)\delta_j(y) = \sum_{k+r=n} \gamma_k(x')\delta_r(y')$$

Hence ϵ_n is well-defined and we only need to check that the axioms hold. Axioms (DP1) and (DP3) follow by a straightforward computation. The case of (DP4) is only slightly harder:

$$\epsilon_n((x+y)+(x'+y')) = \sum_{i+j=n} \gamma_i(x+x')\delta_j(y+y') = \sum_{k+l+r+s=n} \gamma_k(x)\gamma_l(x')\delta_r(y)\delta_s(y') = \sum_{a+b=n} \epsilon_a(x+y)\epsilon_b(x'+y').$$

Finally, to check (DP2) and (DP5) it is enough to do it when $x = 0$ or $y = 0$ by Lemma 2.14. Both cases are obvious. $\square$

5. Flat extensions of divided power rings

The next result will allow to globalise the notion of divided power structure to schemes among other things.

LEMMA 2.16. *Let (R, I, γ) be a divided power ring. Let S be a flat R-algebra. Then there is a unique extension of γ to (S, IS).*

PROOF. Suppose that γ extends to $\overline{\gamma}$ on (S, IS). Let $s \in IS$. Write $s = \sum_{i=1}^r x_i s_i$ with $x_i \in I$, $s_i \in S$. Then $\overline{\gamma}$ has to be given by the formula

$$\overline{\gamma}_n(s) = \sum_{d_1+\cdots+d_r=n} s_1^{d_1}\cdots s_r^{d_r}\gamma_{d_1}(x_1)\ldots\gamma_{d_r}(x_r). \tag{1}$$

[2]We compute in $\mathbb{Z}[T]\langle X,Y\rangle$:

$$\gamma_n(\gamma_m(TX+Y)) = \gamma_n(\sum_{i+j=m}\gamma_i(TX)\gamma_j(Y)) = \sum_{e_0+\cdots+e_m=n}\prod_{i=0}^m \gamma_{e_i}(\gamma_i(TX)\gamma_{m-i}(Y))$$
$$= \sum_{e_0+\cdots+e_m=n}\prod_{i=0}^m T^{ie_i}\gamma_i(X)^{e_i}\gamma_{e_i}(\gamma_{m-i}(Y)).$$

Applying φ and running the same computation backwards shows what we need.

This shows uniqueness. Next we will show that the above expression is independent of the particular choice of x_i and s_i. Suppose first that we have $c_1, \dots, c_N \in S$ and $a_{ij} \in R$ such that $s_i = \sum_{j=1}^N a_{ij} c_j$. Then $s = \sum x_i s_i = \sum_{j=1}^N (\sum_{i=1}^r x_i a_{ij}) c_j$. Let $y_j = \sum_{i=1}^r x_i a_{ij} \in I$. We claim that

$$(2) \qquad \sum_{e_1 + \dots + e_r = n} s_1^{e_1} \cdots s_r^{e_r} \gamma_{e_1}(x_1) \cdots \gamma_{e_r}(x_r) = \sum_{d_1 + \dots + d_N = n} c_1^{d_1} \cdots c_N^{d_N} \gamma_{d_1}(y_1) \cdots \gamma_{d_N}(y_N).$$

Indeed, the divided powers of the x_i define a homomorphism

$$\varphi : \mathbb{Z}[A_{ij}, C_j] \langle X_1, \dots, X_r \rangle \to A$$

such that $\varphi(A_{ij}) = a_{ij}$, $\varphi(C_j) = c_j$, $\varphi(X_i^{[n]}) = \gamma_n(x_i)$. Put $Y_j = \sum_{i=1}^r A_{ij} X_i$, $B_i = \sum_{j=1}^N A_{ij} C_J$. Then it is easy to see[3] that $\varphi(\gamma_d(Y_j)) = \gamma_d(y_j)$. It follows that it is enough to check identity (2) in $\mathbb{Z}[A_{ij}, C_j] \langle X_1, \dots, X_r \rangle$, where it is obvious because $\overline{\gamma}$ is a divided power structure.

With this identity at our disposal we can finish proving that the expression in (1) is well-defined. Suppose that $s \in IS$ can be written in two different ways: $s = \sum_i x_i s_i = \sum_i x_i' s_i'$ for some $x_i, x_i' \in I$, $s_i, s_i' \in S$. By Lazard's theorem[4] we can find $c_j \in S$, $a_{ij}, a_{ij}' \in R$ such that $s_i = \sum a_{ij} c_j$, $s_i' = \sum a_{ij}' c_j$ and such that $y_j = \sum_i a_{ij} x_i = \sum_i a_{ij}' x_i' = y_j'$. Equation (2) shows that the expression (1) using the x_i's and s_i's coincides with the same expression using the x_i''s and s_i''s.

Finally, we need to check the axioms. Axioms (DP1), (DP3) and (DP4) follow directly from the definition. Axioms (DP2) and (DP5) can be checked on generators $s \in I$ by Lemma 2.14. Since $\overline{\gamma} = \gamma$ on I, we are done. □

Using this result we can extend our construction of divided power polynomial algebras so that the resulting divided power structure is compatible with the divided power structure on the base ring A (if there is any).

COROLLARY 2.17. *Let (R, I, γ) be a divided power ring. Let Λ be a set and let $J = IR\langle X_\lambda : \lambda \in \Lambda \rangle + R\langle X_\lambda : \lambda \in \Lambda \rangle_+$. Then there is a unique divided power structure on $(R\langle X_\lambda : \lambda \in \Lambda \rangle, J)$ compatible with the divided power structure on (R, I) and the one on $(R\langle X_\lambda : \lambda \in \Lambda \rangle, R\langle X_\lambda : \lambda \in \Lambda \rangle_+)$.*

PROOF. Observe that $IR\langle X_\lambda : \lambda \in \Lambda \rangle \cap R\langle X_\lambda : \lambda \in \Lambda \rangle_+ = IR\langle X_\lambda : \lambda \in \Lambda \rangle_+$. Thus it follows from Lemma 2.15 that it is enough to show that γ extends to a divided power structure on $(R\langle X_\lambda : \lambda \in \Lambda \rangle, IR\langle X_\lambda : \lambda \in \Lambda \rangle)$, which follows from Lemma 2.16, since $R \to R\langle X_\lambda : \lambda \in R \rangle$ is a free (hence flat) extension. □

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[3] Compute $\gamma_{d_j}(Y_j) = \sum_{e_1 + \dots + e_r = d_j} \prod_{i=1}^r A_{ij}^{e_i} \gamma_{e_i}(X_i)$. Applying φ and doing the same calculation backwards gives the desired identity.

[4] Lazard's Theorem: *Every flat R-module is the direct limit of finite free R-modules* [7, Tag 058G]. Then there is a finite free R-module F together with a homomorphism of R-modules $\psi : F \to S$ such that the *situation* s, s_i, s_i', x_i, x_i' arises from a similar situation on F via ψ: there exists $\beta, \beta_i, \beta_i', \xi_i, \xi_i' \in F$ such that $\beta = \sum_i \beta_i \xi_i = \sum_i \beta_i' \xi_i'$ and $\psi(\beta) = s$, $\psi(\beta_i) = s_i$, $\psi(\beta_i') = s_i'$, $\psi(\xi_i) = x_i$, $\psi(\xi_i') = x_i'$. Let $\gamma_j \in F$ be an R-basis. Then there exists $\alpha_{ij}, \alpha_{ij}' \in R$ such that $\beta_i = \sum_j \alpha_{ij} \gamma_j$, $\beta_i' = \sum_j \alpha_{ij}' \gamma_j$. Then $\beta = \sum_j \nu_j \gamma_j = \sum_j \nu_j' \gamma_j$, where $\nu_j = \sum_i \alpha_{ij} \xi_i$, $\nu_j' = \sum_i \alpha_{ij}' \xi_i'$. Since the γ_j are a basis, we deduce that $\nu_j = \nu_j'$ for all j. Let $c_j = \psi(\gamma_j)$, $a_{ij} = \psi(\alpha_{ij})$, $a_{ij}' = \psi(\alpha_{ij}')$. Then $s = \psi(\beta) = \sum_j y_j c_j = \sum_i x_i s_i = \sum_i x_i' s_i'$, where $s_i = \sum_j a_{ij} c_j$, $s_i' = \sum_j a_{ij}' c_j$, $y_j = \psi(\nu_j)$, $y_j' = \psi(\nu_j')$ so that $y_j = \sum_i a_{ij} x_i = \sum_i a_{ij}' x_i' = y_j'$.

REMARK 2.18. With the above definition, we have a natural isomorphism of divided power rings $R\langle X,Y\rangle = R\langle X\rangle\langle Y\rangle$.

REMARK 2.19. We haven't used (DP5) in any meaningful way in the proofs of this chapter (it is only used to prove things about (DP5) itself). However, in the next chapter it will be used at a crucial point when proving the existence of divided power envelopes.

6. Divided power differential forms

Let (S,J,δ) a divided power ring. Let $R \to S$ a ring homomorphism (we say that (S,J,δ) is a *divided power algebra over* R). We define the *module of divided power Kähler differentials* $\Omega^1_{S/R,\delta} = \Omega^1_{(S,J,\delta)/R}$ of (S,J,δ) over R by taking the quotient of $\Omega^1_{S/R}$ by the relation

$$d\delta_n(b) = \delta_{n-1}(b)db \quad b \in J, n \geq 1.$$

EXERCISE 2.20. Let M be a S-module. A *divided power R-derivation* into M is a map $\theta : S \to M$ which is additive, satisfies $\theta(xx') = x'\theta(x) + x\theta(x')$ for all $x, x' \in S$ and $\theta(r) = 1$ for all $r \in R$, and such that $\theta(\gamma_n(x)) = \gamma_{n-1}(x)\theta(x)$ for all $x \in J$, $n \geq 1$. Show that the natural map $d : S \to \Omega^1_{S/R,\delta}$ is the universal divided power R-derivation. That is, given any θ as above, there exists a unique S-linear map $u : \Omega^1_{S/R,\delta} \to M$ such that $ud = \theta$. Deduce that $\Omega^1_{D_S(J)/R,\delta} = \Omega^1_{S/R} \otimes_S D_S(J)$.

6.1. Divided power de Rham complex. Let (S,J,δ) be a divided power algebra over R. The *de Rham complex of* (S,J,δ) *over* R is defined by $\Omega^n_{S/R,\delta} = \bigwedge^n \Omega^1_{S/R,\delta}$:

$$S \to \Omega^1_{S/R,\delta} \to \Omega^2_{S/R,\delta} \to \cdots \to \Omega^n_{S/R,\delta} \to \cdots$$

There is an interesting filtration on this complex, called the *divided power Hodge filtration:*

$$\mathrm{Fil}^n\,\Omega^\bullet_{S/R,\delta} = J^{[n]} \to J^{[n-1]}\Omega^1_{S/R,\delta} \to \cdots \to J\Omega^{n-1}_{S/R,\delta} \to \Omega^n_{S/R,\delta} \to \cdots$$

REMARK 2.21. Let (R,I,γ) be a divided power ring. Let $R \to B$ be a flat homomorphism. Then $D_{B,\gamma}(IB) = B$ equipped with the unique extension of γ to (B,IB). Furthermore, $\Omega^1_{B/R,\delta} = \Omega^1_{B/R}$ (usual differential forms). The first assertion is clear from the universal property. The second one follows from observing that the relation $d\gamma_n(b) = \gamma_{n-1}(b)db$ is automatic if $b \in IB$:

$$\begin{aligned} d(\gamma_n(z_1+z_2)) &= d\left(\sum_{i+j=n}\gamma_i(z_1)\gamma_j(z_2)\right) \\ &= \sum_{i+j=n} d(\gamma_i(z_1)\gamma_j(z_2)) \\ &= \sum_{i+j=n} \gamma_j(z_2)\gamma_{i-1}(z_1)dz_1 + \gamma_i(z_1)\gamma_{j-1}(z_2)dz_2 \\ &= \sum_{r+s=n-1} \gamma_r(z_1)\gamma_s(z_2)(dz_1+dz_2) \\ &= \gamma_{n-1}(z_1+z_2)d(z_1+z_2). \end{aligned}$$

Thus by induction it is enough to check the identity $d\gamma_n(xb) = \gamma_{n-1}(xb)d(xb)$ for $x \in I$, $b \in B$.

$$\begin{aligned} d\gamma_n(xb) &= d(b^n\gamma_n(x)) = \gamma_n(x)nb^{n-1}db \\ &= \gamma_{n-1}(x)xb^{n-1}db = \gamma_{n-1}(xb)d(xb). \end{aligned}$$

Lemma 2.22. *Let (B, J, δ) be a divided power algebra over R. Consider $B\langle X\rangle$ with divided power structure $(JB\langle X\rangle + B\langle X\rangle_+, \delta')$. Then there is a natural isomorphism of $B\langle X\rangle$-modules*

$$\Omega^1_{B\langle X\rangle/R,\delta'} \simeq \Omega^1_{B/R,\delta} \otimes_B B\langle X\rangle \oplus \Omega^1_{B\langle X\rangle/B,\delta'}.$$

In particular, we obtain natural filtered *isomorphisms*

$$\Omega^\bullet_{B\langle X\rangle/R,\delta'} \simeq \Omega^\bullet_{B/R,\delta} \otimes_B B\langle X\rangle \otimes_{B\langle X\rangle} \Omega^\bullet_{B\langle X\rangle/B,\delta'}.$$

Proof. We begin by observing that the section $B \to B\langle X\rangle$ to the projection $B\langle X\rangle \to B$ implies that the sequence

$$0 \to \Omega^1_{B/R} \to \Omega^1_{B\langle X\rangle/R} \to \Omega^1_{B\langle X\rangle/B} \to 0$$

is split exact. Since the relation imposed corresponds under this maps, we deduce the statement of the Lemma. $\square$

7. Divided power envelope

Theorem 2.23. *Let (R, I, γ) be a divided power ring. The forgetful functor from divided power algebras over (R, I, γ) to algebras over (R, I) that sends (B, J, δ) to (B, J) has a left adjoint $(B, J) \mapsto (D_{B,\gamma}(J), \overline{J}, \overline{\gamma})$.*

$$\begin{tikzcd} & (D_{B,\gamma}(J), \overline{J}, \overline{\gamma}) \arrow[dr, dashed] & \\ (B, J) \arrow[ur] \arrow[rr] & & (C, K, \delta) \\ & (R, I, \gamma) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof. Let B be a ring, $J \subset B$ an ideal. Let $f : (R, I) \to (B, J)$ be the structure homomorphism. Choose a system of generators $J = \langle x_\alpha\rangle_\alpha$ and let L be the kernel of the homomorphism $\pi : \bigoplus_\alpha BX_\alpha \to J$ that sends $\sum_\alpha b_\alpha X_\alpha$ to $\sum_\alpha b_\alpha x_\alpha$. Let $D' = B\langle X_\alpha : \alpha\rangle / L_1$, where L_1 is the divided power ideal generated[5] by L inside $B\langle X_\alpha : \alpha\rangle$. Since L_1 is generated by homogeneous elements, the quotient D' is still a graded B-algebra. Let D'_+ be the image of the ideal $B\langle X_\alpha : \alpha\rangle_+$. Since L_1 is a divided power ideal by construction, the divided power structure on $B\langle X_\alpha : \alpha\rangle_+$ extends uniquely to a divided power structure on D'_+. Let $\varphi : J \to D'_+$ be the map induced by the natural inclusion $\bigoplus_\alpha BX_\alpha \to B\langle X_\alpha : \alpha\rangle$, so that $\varphi(x_\alpha) = X_\alpha^{[1]}$. Consider the ideal L_2 of D' generated by the following two types of elements:

i) $x - \varphi(x)$, $x \in J$.
ii) $\gamma_n(\varphi(f(y))) - f(\gamma_n(y))$ for $y \in I$, $n \geq 0$.

---

[5]This means that L_1 is the ideal generated by $\gamma_e(x)$ for $e \geq 1$ and $x \in L$. Note that since $\gamma_n(\sum_{i=1}^r a_i X_i) = \sum_{e_1+\cdots+e_r=n} \prod_{i=1}^r a_i^{e_i}\gamma_{e_i}(X_i) \in B\langle X_\alpha : \alpha\rangle_n$, L_1 is a homogeneous ideal.

Let $D = D'/L_2$. We check that L_2 is a divided power ideal. We need to check that $L_2 \cap D'_+$ is stable under the divided power operations of D'. For this, let J_1 be the ideal generated by elements of the form $x_\alpha - X_\alpha^{[1]}$ and let J_2 be the ideal generated by elements of the form $\gamma_n(\varphi(f(y))) - \varphi(f(\gamma_n(y)))$ for $y \in I$, $n \geq 0$, so that $L_2 = J_1 + J_2$. Observe that since $J_2 \subset D'_+$ we have $L_2 \cap D'_+ = J_1 \cap D'_+ + J_2$. So it is enough to check that γ sends both $J_1 \cap D'_+$ and J_2 into $L_2 \cap D'_+$.

Let $x \in J_1 \cap D'_+$. Then $x \in J_1$ so we can write $x = \sum_\alpha a_\alpha(x_\alpha - X_\alpha^{[1]})$ for some $a_\alpha \in D'$. Since $x \in D'_+$, looking at the component in degree zero, we see that $\sum_\alpha a_\alpha^0 x_\alpha = 0$ (where we write $a_\alpha = a_\alpha^0 + a_\alpha^+$ with $a_\alpha^0 \in D'_0$, $a_\alpha^+ \in D'_+$). Then also $\sum_\alpha a_\alpha^0 X_\alpha^{[1]} = \sum_\alpha a_\alpha^0 \varphi(x_\alpha) = 0$. Thus $x = \sum_\alpha a_\alpha^+(x_\alpha - X_\alpha^{[1]}) \in J_1 D'_+$. It is immediate that $J_1 D'_+$ is stable under the divided power operations of D'_+, so we are done in this case.

We treat the case of J_2 next. We compute modulo L_2:

$$\begin{aligned}
\gamma_r(\gamma_n(\varphi(f(y))) &- \varphi(f(\gamma_n(y)))) \\
&= \sum_{i+j=r} \gamma_i(\gamma_n(\varphi(f(y))))(-1)^j \gamma_j(\varphi(f(\gamma_n(y)))) \\
&= \sum_{i+j=r} \frac{(in)!}{(n!)^i i!} \gamma_{in}(\varphi(f(y)))(-1)^j \gamma_j(\varphi(f(\gamma_n(y)))) \\
&\equiv \sum_{i+j=r} \frac{(in)!}{(n!)^i i!} f(\gamma_{in}(y))(-1)^j f(\gamma_j(\gamma_n(y))) \\
&= f\left(\sum_{i+j=r} \frac{(in)!}{(n!)^i i!} \gamma_{in}(y)(-1)^j \gamma_j(\gamma_n(y))\right) \\
&= f\left(\sum_{i+j=r} \gamma_i(\gamma_n(y))\gamma_j(-\gamma_n(y))\right) \\
&= f\left(\gamma_r(\gamma_n(y) - \gamma_n(y))\right) = 0.
\end{aligned}$$

Thus, the ideal L_2 is a divided power ideal of D' and so the image $\overline{J}$ of D'_+ in D carries a uniquely determined divided power structure $\overline{\gamma}$.

Property i) above shows that $JD \subset \overline{J}$. Property ii) shows that $\overline{\gamma}$ is compatible with γ. Now suppose that $g : (B, J) \to (C, K, \delta)$ is a homomorphism over (A, I, γ). Then the $g(x_\alpha) \in K$ determine a homormorphism $B\langle X_\alpha : \alpha\rangle \to C$. It is easy to see that this homomorphism factors through D. $\square$

Exercise 2.24. Let (R, I, γ) be a divided power ring. Let $J = IR[X] + XR[X]$. Then there is a canonical isomorphism $D_{R[X],\gamma}(J) \simeq R\langle X\rangle$ of divided power rings.

There are a number of immediate consequences of the above construction. In the following remarks, we keep the notation of the statement of the theorem.

Remark 2.25. The natural map $B/J \to D_{B,\gamma}(J)/\overline{J}$ is an isomorphism. Indeed, B/J has a trivial divided power structure (on the zero ideal) so by universality there exists a map $D_{B,\gamma}(J) \to B/J$ sending $\overline{J}$ to zero. This is the inverse of the above map.

REMARK 2.26. Suppose that $(R, I, \gamma) \to (R', I', \gamma')$ is a surjective homomorphism. Let $B' = R' \otimes_R B$, $J' = JB'$. The the canonical map $R' \otimes_R D_{B,\gamma}(J) \to D_{B',\gamma}(J')$ is an isomorphism. This is immediate from the universal property.

REMARK 2.27. Suppose that there is an ideal $K \subset B$ such that $KD_{B,\gamma}(J) = 0$. Then $D_{B,\gamma}(J) \simeq D_{B/K,\gamma}(J/J \cap K)$. This is immediate from the universal property.

REMARK 2.28. Let $B \to C$ be a flat homomorphism. Then $D_{B,\gamma}(J) \otimes_B C$ is a divided power ring (Lemma 2.16) and the natural map $D_{B,\gamma}(J) \otimes_B C \to D_{C,\gamma}(JC)$ is an isomorphism of divided power rings. This is immediate from the universal property (use that because $B \to C$ is flat, $\overline{J}(D_{B,\gamma}(J) \otimes_B C) = \overline{J} \otimes_B C$).

EXERCISE 2.29. Let $B \to C$ be a ring homomorphism. Suppose that the natural map $D_{B,\gamma}(J) \otimes_B C \to D_{C,\gamma}(JC)$ is an isomorphism. Then every divided power structure on (B, J) extends to a divided power structure on (C, JC). Deduce that divided power structures extend over flat extensions.

REMARK 2.30. Let (R, I, γ) be a divided power ring. Let $(R, I) \to (B, J_1)$ be a homomorphism. Let $J \subset B$ be an ideal containing J_1. Let $(D_B(J_1), \overline{J}_1, \overline{\gamma}_1)$ be the divided power envelope of (B, J_1). Then there is a canonical isomorphism

$$D_{D_{B,\gamma}(J_1),\overline{\gamma}_1}(JD_B(J_1) + \overline{J}_1) \simeq D_{B,\gamma}(J)$$

of divided power rings.

8. Divided power envelopes and smooth embeddings

THEOREM 2.31. *Let (R, I, γ) be a divided power ideal. Let $R \to C$ be a homomorphism. Let $J \subset K \subset C$ be ideals such that J is generated by a regular sequence $x_1, \dots, x_d \in C$. Suppose that $R \to C$ and $R \to C/J$ are smooth.*

$$\begin{tikzcd} C/K & C/J \arrow[l, two heads] & C \arrow[l, two heads] \\ & R \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Then there is an isomorphism $D_{C,\gamma}(K) \simeq D_{C/J,\gamma}(KC/J)\langle X_1, \dots, X_d\rangle$ of divided power rings making the following diagram commute:

$$\begin{tikzcd} D_{C,\gamma}(K) \arrow[rr, "\simeq"] \arrow[dr, two heads] & & D_{C/J,\gamma}(KC/J)\langle X_1, \dots, X_d\rangle \arrow[dl, two heads, "X_i^{[n]} \mapsto 0, n>0"] \\ & D_{C/J,\gamma}(KC/J) & \end{tikzcd}$$

In order to prove this theorem we need a lemma (which is the particular case of the theorem when K and J coincide).

LEMMA 2.32. *There is a canonical isomorphism*

$$D_{C,\gamma}(J) \simeq D_{C/J[X_1,\dots,X_d],\gamma}(J_0)$$

of divided power rings.

PROOF. Since p is nilpotent on R, we see that $J^N D_{C,\gamma}(J) = 0$ for some $N > 0$ (Remark 2.27). We have a commutative diagram

$$\begin{tikzcd}
D_{C,\gamma}(J) \arrow[d, "\simeq"] & D_{B/J[X_1,\ldots,X_d],\gamma}(J_0) \arrow[l, dashed, "\simeq"'] \arrow[d, "\simeq"] \\
D_{C/J^N,\gamma}(J/J^N) & D_{B/J[X_1,\ldots,X_d]/J_0^N,\gamma}(J_0/J_0^N) \arrow[l, "\simeq"'] \\
C/J^N \arrow[u] & C/J[X_1,\ldots,X_d]/J_0^N \arrow[l, "\simeq"'] \arrow[u]
\end{tikzcd}$$

Then we are done. □

REMARK 2.33. Since γ extends uniquely to C/J by flatness, we see that

$$D_{C/J[X_1,\ldots,X_d],\gamma}(J_0) \simeq C/J\langle X_1,\ldots,X_d\rangle$$

as divided power rings.

PROOF OF THEOREM 2.31. We have a sequence of isomorphisms

$$\begin{aligned} D_{C,\gamma}(K) &\simeq D_{D_{C,\gamma}(J),\overline{\gamma}_1}(KD_{C,\gamma}(J) + \overline{J}_1) \\ &\simeq D_{D_{C/J[X_1,\ldots,X_d],\gamma}(J_0),\overline{\gamma}_1}(K + \overline{J}_1) \\ &\simeq D_{C/J[X_1,\ldots,X_d],\gamma}(J_0 + KC/J[X_1,\ldots,X_d]) \\ &\simeq D_{C/J,\gamma}(K)\langle X_1,\ldots,X_d\rangle . \end{aligned}$$

The first isomorphism comes from Lemma 2.30 with $J \subset K$, the second one from Lemma 2.32, the third isomorphism again from Lemma 2.30 with $J_0 \subset J_0 + KC/J[X_1,\ldots,X_d]$ and the fourth from a simple comparison of universal properties. □

THEOREM 2.34. *Let $J \subset I$ a divided power subideal. If X is smooth over R/J, then $\mathrm{Fil}^n\,\Omega^\bullet_{D_Y(X)}$ is flat over R. If X is smooth over R, let $X' = X \otimes R/J$, $Y' = Y \otimes R/J$. Then the natural maps*

$$\mathrm{Fil}^n\,\Omega^\bullet_{D_Y(X)} \otimes_R R/J \to \mathrm{Fil}^n\,\Omega^\bullet_{D_{Y'}(X')}$$

are isomorphisms.

PROOF. This statement is local on Y_m. We can assume that X_0, Y_m is affine and even that $X_0 = \operatorname{Spec} A$ is of the form

$$A = R/I[X_1,\ldots,X_d]/(f_1,\ldots,f_r)$$

where $\det(\partial f_i/\partial X_j)_{i,j=1,\ldots,r} \in A^\times$ (Corollary A.20). Let $g_1,\ldots,g_r \in R[X_1,\ldots,X_d]$ some lift of the $f_1,\ldots,f_r$. Then $B = R[X_1,\ldots,X_d]/(g_1,\ldots,g_r)$ is a smooth R-algebra (because $\det(\partial g_i/\partial X_j)$ is a unit in B/I, and therefore it is a unit in B[6]). By construction we have $B \otimes_R R/I = A$. Furthermore, smoothness of Y_m implies that there is a map $\operatorname{Spec} B \to Y_m$ extending the closed embedding $X_0 \to Y_m$. It is

[6] Let us show that the if $a \in A$ is a unit in A/I and $I^2 = 0$, then it is a unit in A. Suppose that $a \in A$ is such that $a \in (A/I)^\times$. Then there exists $x \in A$ such that $ax \equiv 1 \pmod I$. That is, $ax - 1 = z \in I$. Squaring, we get $x(-a^2x + 2a) - 1 = -a^2x^2 + 2ax - 1 = -z^2 = 0$, so that $x \in A^\times$ and we are done.

easy to see that any such map must be a closed embedding[7]. Thus we can assume that there is a diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, hook, "closed"] \arrow[d] & Y_m \arrow[dl] \\ \operatorname{Spec} R/I \arrow[r, hook] & \operatorname{Spec} R & \end{tikzcd}$$

where the square is cartesian, X_0 is smooth over R/I, and X and Y_m are smooth over R. By Theorem 2.31 and Remark 2.21, It follows that locally we have an isomorphism $D_{Y_m}(X_0) = B\langle X_1, \ldots, X_d\rangle$, where $X = \operatorname{Spec} B$. Lemma 2.22 shows that in this case the module of Kähler differential is free over B and thus, locally free over R. $\square$

THEOREM 2.35. *Let (R, I, γ) be a divided power ring. Let $R \to B \to C$ be ring homomorphisms. Suppose that $B \to C$ is etale. Let $J \subset B$, $K \subset C$ be ideals such that $B/J = C/K$. Then there are compatible isomorphisms $B/J^N \simeq C/K^N$, $N = 1, 2, \ldots$, lifting the identification $B/J = C/K$, under which the ideal JB/J^N corresponds to the ideal KC/K^N. In particular, $D_{B,\gamma}(J) \simeq D_{C,\gamma}(K)$ as divided power rings.*

PROOF. By smoothness of B, there are (unique) maps $B \to C/K^N$ for every N lifting the map $B \to B/J = C/K$. Clearly this maps factors through B/J^N. Conversely we get a map $C/K^N \to B/J^N$. By uniqueness we show that they are isomorphisms. The claim on the ideals is obvious. To see the claim about divided power envelopes, use Remark 2.27. $\square$

[7]Let us show that if R is a ring, $I \subset R$ is an ideal, $I^2 = 0$, $\phi : A \to B$ is a map of R-algebras such that $A \to B \to B/IB$ is surjective, then $A \to B$ is surjective. Let $b \in B$. Then there exists $a' \in A$ such that $\phi(a') = b + b'$ for some $b' \in IB$. Write $b' = \sum x_i b_i$ for some $x_i \in I$, $b_i \in B$. Repeating the argument we write $b_i + b_i' = \phi(a_i)$ for some $a_i \in A$, $b_i' \in IB$. Then $x_i b_i' = 0$ so that we get that $\phi(a') = b + \sum x_i \phi(a_i)$, i.e., $\phi(a' - \sum x_i a_i) = b$ and we are done.

CHAPTER 3

Crystalline cohomology

Fix a prime number p. Let (R, I, γ) be a divided power ring. We assume that R is a $\mathbb{Z}/m$-algebra for some non-zero $m \in \mathbb{Z}$. Let $R \to C$ be a ring homomorphism and let $J \subset K \subset C$ be ideals. Suppose that $R \to C$ and $R \to C/J$ are smooth.

$$\begin{tikzcd} \operatorname{Spec} D_{C/J,\gamma}(KC/J) \arrow[r, hook] \arrow[d] & \operatorname{Spec} D_{C,\gamma}(K) \arrow[d] \\ \operatorname{Spec} C/J \arrow[r, hook] & \operatorname{Spec} C \end{tikzcd}$$

Our objective is to show that the natural map from the divided power de Rham complex of $D_{C,\gamma}(K)$ to that of $D_{C/J,\gamma}(KC/J)$ is a (filtered) quasi-isomorphism. Note that here the divided power envelope is playing the role of the K-adic completion in the characteristic zero situation.

0.1. Things to explain. The most important cases are when $K = J$ and when $K = J + I$. Also explain that we can always mod out by I, but that the filtration changes, and that the filtration of something smooth coincides with the Hodge filtration.

1. Poincaré lemma

Theorem 3.1 (Poincaré Lemma). *Let (B, J, δ) be a divided power ring over R. Then the natural map*

$$z : \Omega^\bullet_{B\langle X_1,\ldots,X_d\rangle/R,\delta} \to \Omega^\bullet_{B/R,\delta}$$

induced by sending $X_i^{[n]}$ to zero is a filtered homotopy equivalence of R-modules.

Proof. By induction we can assume $d = 1$, $X = X_1$. Let $j : B \to B\langle X\rangle$ be the natural inclusion. Then $zj = 1$. We will show that jz is filtered homotopy equivalent to the identity. We begin by showing that the map

$$B \to \Omega^\bullet_{B\langle X\rangle/B,\delta} = [B\langle X\rangle \to B\langle X\rangle\ dX]$$

is a homotopy equivalence. We use the "integral" $S : B\langle X\rangle\, dX \to B\langle X\rangle$, $X^{[i]}dX \mapsto X^{[i+1]}$ to prove this. We compute $Sd : B\langle X\rangle \to B\langle X\rangle$. Let $b = \sum_{i\geq 0} b_i X^{[i]} \in B\langle X\rangle$.

$$Sd(b) = S\left(\sum_{i\geq 1} b_i X^{[i-1]} dX\right) = \sum_{i\geq 1} b_i X^{[i]} = b - b_0 = b - jz(b).$$

Next we compute $dS : B\langle X\rangle dX \to B\langle X\rangle dX$. Let $\omega = \sum_{i\geq 0} b_i X^{[i]} dX \in B\langle X\rangle dX$,

$$dS(\omega) = d\left(\sum_{i\geq 0} b_i X^{[i+1]}\right) = \sum_{i\geq 0} b_i X^{[i]} = \omega = \omega - jz(\omega).$$

So S gives a homotopy $jz \sim 1$. It follows that the map

$$\Omega^\bullet_{B/R,\delta} \to \Omega^\bullet_{B/R,\delta} \otimes_B \Omega^\bullet_{B\langle X\rangle/R,\delta} \simeq \Omega^\bullet_{B\langle X\rangle/R,\delta}$$

(where the isomorphism comes from Lemma 2.22) is a homotopy equivalence. The only thing left to check is that the homotopy

$$S : \Omega^{n+1}_{B/R,\delta} \otimes_B B\langle X\rangle \oplus \Omega^n_{B/R,\delta} \otimes_B B\langle X\rangle dX \to \Omega^n_{B/R,\delta} \otimes_B B\langle X\rangle \oplus \Omega^{n-1}_{B/R,\delta} \otimes_B B\langle X\rangle dX$$

preserves the filtration. In this direct sum decomposition, S is zero on the first factor and $1 \otimes S$ on the second one. Hence it sends the filtration $J^{[i-n]}\Omega^n_{B/R,\delta} \otimes_B B\langle X\rangle^{[j]}_+ B\langle X\rangle dX$ to $J^{[i-n]}\Omega^i_{B/R,\delta} \otimes B\langle X\rangle^{[j+1]}_+$ and we are done. □

2. A technical lemma

LEMMA 3.2. *Let R be a ring. Let $R \to C \to B$ be smooth homomorphisms. Let $J \subset B$, $K \subset C$ be ideals such that $B/J = C/K = A$. Let $x \in \operatorname{Spec} A$. Then there exists $f \in B$ and an ideal $J_0 \subset J_f$ such that $f(x) \neq 0$ and $C \to B_f/J_0$ is etale.*

$$\begin{tikzcd}
\operatorname{Spec} B_f/J_0 \arrow[hookrightarrow, d] & \operatorname{Spec} A_f \ni x \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec} B_f \arrow[r] \arrow[d] & \operatorname{Spec} C \arrow[dl] \\
\operatorname{Spec} R &
\end{tikzcd}$$

PROOF. Consider the diagram

$$\begin{tikzcd}
K/K^2 \arrow[r, hook] \arrow[d] & \Omega^1_{C/R} \otimes_C A \arrow[r, two heads] \arrow[d] & \Omega^1_{A/R} \arrow[d, equal] \\
J/J^2 \arrow[r, hook] \arrow[dr, "\beta"'] & \Omega^1_{B/R} \otimes_B A \arrow[r, two heads] \arrow[d, two heads] & \Omega^1_{A/R} \\
& \Omega^1_{B/C} \otimes_B A &
\end{tikzcd}$$

It follows that β is surjective[1]. Let $k = k(x)$. Since $\Omega^1_{B/C}$ is finite projective of rank n, we can find $g_1, \dots, g_n \in I$ such that $dg_1, \dots, dg_n$ is a basis of $\Omega^1_{B_f/C}$ for

[1]Let $x \in \Omega^1_{B/C} \otimes A$. There exists $x_1 \in \Omega^1_{B/R} \otimes A$ projecting down to x. Let $x_2 \in \Omega^1_{C/R} \otimes A$ have the same projection than x_1 in $\Omega^1_{A/R}$. Let x_3 be the projection of x_2 down in $\Omega^1_{B/R} \otimes A$. Then $x_1 - x_3 \in J/J^2$ and $\beta(x_1 - x_3) = \beta(x_1) = x$.

some $f \in B$, $f(x) \neq 0$. Let $J_0 = (g_1, \ldots, g_n) \subset J$. We only need to check that $C \to B/J_0$ is etale. Consider the exact sequence

$$J_0/J_0^2 \to \Omega^1_{B_f/C} \otimes_{B_f} B_f/J_0 \to \Omega^1_{(B_f/J_0)/C} \to 0$$

By construction the first map is surjective. Since $\Omega^1_{B_f/C}$ is a free B_f-module of rank n and J_0 is generated by n elements, it follows that the first map must be an isomorphism. Hence $\Omega^1_{(B_f/J_0)/C} = 0$ and we are done (Theorem A.19). □

3. Definition of crystalline cohomology

Now we extend everything to schemes.

Definition 3.3. Let X be a topological space. Let $\mathscr{O}$ be a sheaf of rings on X and let $\mathscr{I} \subset \mathscr{O}$ be an ideal sheaf. A *divided power structure* on $(\mathscr{O}, \mathscr{I})$ is a sequence of maps $\gamma_n : \mathscr{O} \to \mathscr{O}$, $n = 0, 1, \ldots$, such that the for every $U \subset X$, the maps $\gamma_n(U) : \mathscr{O}(U) \to \mathscr{I}(U)$ are a divided power structure on $(\mathscr{O}(U), \mathscr{I}(U))$.

Let $(X, \mathscr{O}_X, \mathscr{I}, \gamma)$ be a divided power scheme. Let $X \to \operatorname{Spec} R$ be a morphism. Then we define $\Omega^1_{(X,\mathscr{O}_X,\mathscr{I},\gamma)/R}$ to be $\Omega^1_{\mathscr{O}_X/R,\gamma}$. As usual one gets a divided power de Rham complex:

$$\mathscr{O}_X \to \Omega^1_{(\mathscr{O}_X,\mathscr{I},\gamma)/R} \to \Omega^2_{(\mathscr{O}_X,\mathscr{I},\gamma)/R} \to \cdots$$

Definition 3.4. Let $(S, \mathscr{O}_S, \mathscr{I}, \gamma)$ be a divided power scheme. Let $(X, \mathscr{O}_X, \mathscr{J})$ be a scheme over $(S, \mathscr{O}_S, \mathscr{I})$. We define the divided power envelope of X with respcto to $\mathscr{J}$ over S to be $D_X(Y) = \operatorname{Spec}_{\mathscr{O}_X}(D_{\mathscr{O}_X,\gamma}(\mathscr{I}))$. The natural map $D_X(Y) \to X$ is affine.

Then the main theorem:

Theorem 3.5. *Let P and Y be smooth schemes over R. Let $f : Y \to P$ be a map of schemes which is either a closed embedding or smooth and let $g : X \to Y$ be a map such that g and fg are both closed immersions.*

$$\begin{tikzcd}
 & D_{X,\gamma}(Y) \arrow[r] \arrow[d] & D_{X,\gamma}(P) \arrow[d] \\
X \arrow[ur, hook] \arrow[r, hook, "g"] \arrow[dr] & Y \arrow[r, "f"] \arrow[d] & P \arrow[dl] \\
 & \operatorname{Spec} R &
\end{tikzcd}$$

Then the natural map

$$\Omega^\bullet_{D_{X,\gamma}(P)/R} \longrightarrow \Omega^\bullet_{D_{X,\gamma}(Y)/R}$$

is a filtered quasi-isomorphism of of R-modules on X for all n.

Proof. Since this is a local statement, we can assume all the spaces are affine. say $P = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} C/K$. Suppose first that f is a closed embedding. Then $B = C/J$ and, localising further if necessary, we can assume that J is generated by a regular sequence. Then by Theorem 2.31, we have $D_C(K) = D_{C/J}(KC/J)\langle X_1, \ldots, X_d\rangle$ for $d = \dim P - \dim Y$, and the map

$D_C(K) \to D_{C/J}(KC/J)$ is defined by sending the $X_i^{[n]}$ to zero. Now the conclusion follows from Theorem 3.1. On the other hand, if f is a smooth map, then by Lemma 3.2 we can assume, localising if necessary, that there is an ideal $J_0 \subset J \subset B$ such that $C \to B/J_0$ is etale (in particular, $R \to B/J_0$ is smooth). Then we have a commutative diagram

$$\begin{tikzcd}
\Omega^\bullet_{D_C(K)} \arrow[rr] \arrow[dr] & & \Omega^\bullet_{D_B(J)} \arrow[dl] \\
& \Omega^\bullet_{D_{B/J_0}(JB/J_0)} &
\end{tikzcd}$$

The left diagonal map is a filtered quasi-isomorphism by Theorem 2.35. and the right diagonal map is a filtered quasi-isomorphism by the previous case. ☐

Let X be a scheme over R/I. Let Y, Z smooth schemes over R such that X is a closed subscheme of both Y and Z over R. Consider the product $Y \times_R Z$. It is a smooth scheme over R and Z is a closed subscheme over R. There is a diagram

$$\begin{tikzcd}
& \Omega^\bullet_{D_{X\times Y}(Z)} & \\
\Omega^\bullet_{D_X(Z)} \arrow[ur] & & \Omega^\bullet_{D_Y(Z)} \arrow[ul]
\end{tikzcd}$$

Each of the two maps is a filtered quasi-isomorphism by Theorem 3.5. We check a "cocycle condition": given a third smooth scheme X_3 over R such that Z is a closed subscheme over R, $J_{123} = \ker(B_1 \otimes B_2 \otimes B_3 \to A)$. We have a commutative diagram

$$\begin{tikzcd}
& \Omega^\bullet_{D_{X_1\times X_2\times X_3}(Z)} & \\
\Omega^\bullet_{D_{X_1\times X_2}(Z)} \arrow[ur] & \Omega^\bullet_{D_{X_1\times X_3}(Z)} \arrow[u] & \Omega^\bullet_{D_{X_2\times X_3}(Z)} \arrow[ul] \\
\Omega^\bullet_{D_{X_1}(Z)} \arrow[u] \arrow[ur] & \Omega^\bullet_{D_{X_2}(Z)} \arrow[ul] \arrow[ur] & \Omega^\bullet_{D_{X_3}(Z)} \arrow[u] \arrow[ul]
\end{tikzcd} \tag{3}$$

of filtered quasi-isomorphisms.

Let X, X' be schemes over R/I. Let Y, Y' be smooth schemes over R such that X (resp. X') is a closed subscheme of Y (resp. Y') over R. Suppose that $f : X \to X'$ is a morphism of schemes over R/I and suppose that $g : Y \to Y'$ lifts the map f

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "f"] & Y \arrow[d, "g"] \\
X' \arrow[r, hook] & Y'
\end{tikzcd}$$

Then there is an induced map

$$f^{-1}\Omega^\bullet_{D_{Y'}(X')} \to \Omega^\bullet_{D_Y(X)}$$

We want to express the fact that this map is "independent" of the particular choice of lift g, and only depends on f. Indeed, suppose that Y_2, Y_2' are another set of smooth embeddings of X, X'. Suppose there is a lift $g_2 : Y_2 \to Y_2'$ of the map $f : X \to X'$. Then there is a commutative diagram

$$\begin{tikzcd} p_1^{-1}\Omega^\bullet_{D_{Y_1}(X)} \arrow[r] \arrow[d] & \Omega^\bullet_{D_{Y_1\times Y_2}(X)} \arrow[d] & p_2^{-1}\Omega^\bullet_{D_{Y_2}(X)} \arrow[l] \arrow[d] \\ p_1^{-1}\Omega^\bullet_{D_{Y_1'}(X')} \arrow[r] & \Omega^\bullet_{D_{Y_1'\times Y_2'}(X')} & p_2^{-1}\Omega^\bullet_{D_{Y_2'}(X')} \arrow[l] \end{tikzcd}$$

Moreover, there is always at least one such lift. Indeed, let $f : X \to X'$ be the given map. Let Y, Y' be smooth embeddings of X, X' Let $h : X \to Y$, $g : X' \to Y'$ be the inclusions.

$$\begin{tikzcd} X \arrow[r, hook, "(h,gf)"] \arrow[d, "f"] & Y \times_R Y' \arrow[d, "p_2"] \\ X' \arrow[r, hook, "g"] & Y' \end{tikzcd}$$

DEFINITION 3.6. Let (R, I, γ) be a DP-ring. Let X be a quasi-projective scheme over R/I. Define the crystalline cohomology groups of X with respect to (R, I, γ) as the hypercohomology groups

$$H^*_{crys}(X/R) = \mathbb{H}^*(X, \Omega^\bullet_{D_P(X)/R}).$$

We see that this defines a functor using the diagram (3).

3.1. How the game is going to be played. We are going to be interested in the following situation. X will be a smooth, quasi-projective scheme over R, $X_0 = X \otimes R/I$ and $X \subset P$ will be a an immersion into an open subset of projective scheme. Moreover, we will be given an R-endomorphism of $F_0 : X_0 \to X_0$ equipped with a lift $F : P \to P$.

We begin with the following crucial observation. By Remark 2.21, the crystalline cohomology of X_0 computes the usual algebraic de Rham cohomology of X/R:

$$H^*_{cr}(X_0/R) = H^*_{dR}(X/R).$$

In particular the endomoprphism $F_0 : X_0 \to X_0$ induces an endomorphism $F_0 : H_{dR}(X/R) \to H_{dR}(X/R)$ *even though* F_0 *does not lift to* X *in general.* To understand this action, we bring P into the picture. So suppose that X itself admits an embedding into a smooth P/R that is equipped with a lift $F : P \to P$ of F_0. In practice, P will be projective space or an opan subset of such. Then we proved that we have canonical identifications

$$H_{cr}(X_0/R) = \mathbb{H}^*(X_0, \Omega^\bullet_{D_P(X_0)})$$

such that the action of F_0 on the left corresponds to the action of F on the right. By composition we see that the action of F_0 on $H_{dR}(X/R)$ corresponds to the action of F on $\mathbb{H}(X_0, \Omega^*_{D_P(X_0)})$. The beauty of all this is that the action of F can be written in terms of complexes because F exists on the level of complexes! This will be the key that will allow us to "compute" such actions.

There is one more point to understand about these identifications. By construction we know that there is an identification (even an equality)

$$D_P(X_0) = D_P(X)$$

of PD-rings. However, the filtration in each one of them is different.

4. Čech descent

Here we explain the statement of cohomological descent and the particular case of Čech descent. This can be used to eliminate the hypothesis of quasi-projectivity found in many of the results of this notes.

Let (R, I, γ) be a divided power ring killed by some non-zero integer. Let X be a separated, quasi-compact scheme over R. We consider a *system of local embeddings* $\mathscr{U} = \{U_i, X_i\}$, where $\{U_i\}$ is an affine open cover of X, and for each i, $U_i \to X_i$ is a closed immersion into a affine smooth scheme X_i over R. For each $(p+1)$-tuple $I = \{i_0 < i_1 < \cdots < i_p\}$, we consider the open set

$$U_I = U_{i_0} \cap \cdots \cap U_{i_p}$$

(which is affine since X is separated over R/I) and the embedding

$$U_I \to X_I = X_{i_0} \times_R X_{i_1} \times_R \cdots \times_R X_{i_p}$$

(X_I is a smooth scheme over R.) Then we consider the complex of sheaves on X

$$j_{I*}\Omega^\bullet_{(U_I, X_I)/R}$$

where $j_I : U_I \to X$ is the natural inclusion. Let $I' = I \setminus \{i_j\}$ for some $0 \leq j \leq p$. Then there is a natural inclusion $U_I \to U_{I'}$ and a projection $X_I \to X_{I'}$, which is a smooth morphism. We deduce a map

$$\Omega^\bullet_{(U_{I'}, X_{I'})}\Big|_{U_I} \to \Omega^\bullet_{(U_I, X_I)}$$

Hence we have a map of complexes

$$\delta_{I,j} : j_{I'*}\Omega^\bullet_{D_{X_{I'}}(U_{I'})} \to j_{I*}\Omega^\bullet_{D_{X_I}(U_I)}$$

Define a double complex $\Omega^\bullet_{\mathscr{U}}$ by

$$\Omega^{\bullet,p}_{\mathscr{U}} = \prod_{|I|=p+1} j_{I*}\Omega^\bullet_{D_{X_I}(U_I)}, d^{p-1} = \prod (-1)^j \delta_{I,j}.$$

We use the same notation for the simple complex associated to $\Omega^\bullet_{\mathscr{U}}$.

THEOREM 3.7. *Let $\mathscr{U}$, $\mathscr{V}$ two systems of local embeddings. Then $\mathscr{U} \cap \mathscr{V}$ is a common refinement of $\mathscr{U}$ and $\mathscr{V}$ and the maps*

$$\begin{tikzcd} & \Omega^\bullet_{\mathscr{U}\cap\mathscr{V}} & \\ \Omega^\bullet_{\mathscr{U}} \arrow[ur] & & \mathscr{C}(\mathscr{V}) \arrow[ul] \end{tikzcd}$$

are filtered quasi-isomorphisms of R-modules. They satisfy a "cocycle" condition.

Proof. To show that this definition is independent of the system of local embeddings, we proceed as follows. A *refinement* of the system of local embeddings $\mathscr{U} = \{U_i, X_i\}_{i\in I}$ is another such system $\mathscr{V} = \{V_j, Z_j\}_{j\in J}$, together with a mapping $\lambda : J \to I$ of index sets such that V_j is an open subset of $U_{\lambda(j)}$ for each j, and together with a smooth morphism $Z_j \to X_{\lambda(j)}$ compatible with these inclusions for each j.

$$\begin{tikzcd} V_j \arrow[r, hook] \arrow[d, hook] & Z_j \arrow[d] \\ U_{\lambda(j)} \arrow[r, hook] & X_{\lambda(j)} \end{tikzcd}$$

If $\mathscr{V}$ is a refinement of $\mathscr{U}$, there is a natural map of double complexes

$$\Omega^\bullet_{\mathscr{U}} \to \Omega^\bullet_{\mathscr{V}}$$

Since given any two systems $\mathscr{U}$ and $\mathscr{V}$, the system $\mathscr{U} \times \mathscr{V}$ is a common refinement of both, it is enough to show that for any refinement of $\mathscr{U}$, the associated map of simple complexes is a filtered quasi-isomorphism.

Since the statement is local and $\Omega^\bullet_{\mathscr{U}}|_V = \Omega^\bullet_{\mathscr{U}\cap V}$ (with obvious notation), we can assume that $X = U_1 = V_1$. Let $\mathscr{U}' = \{U_1, X_1\}$, $\mathscr{V}' = \{V_1, Z_1\}$. Then we have a diagram

$$\begin{tikzcd} \Omega^\bullet_{\mathscr{U}'} \arrow[r] \arrow[d] & \mathscr{C}(\mathscr{V}') \arrow[d] \\ \Omega^\bullet_{\mathscr{U}} \arrow[r] & \Omega^\bullet_{\mathscr{V}} \end{tikzcd}$$

The top map is a filtered quasi-isomorphism by Theorem **??**. Hence it is enough to check that the vertical maps are filtered quasi-isomorphisms. Hence we can forget of $\mathscr{V}$. We use the usual spectral sequence

$$H^p_\delta(H^q_d(\Omega^\bullet_{\mathscr{U}})) \implies H^{p+q}(\Omega^\bullet_{\mathscr{U}})$$

Fix p. Then the relevant part of the complex looks like

$$\bigoplus_{|I|=p} H^q(\Omega^\bullet_{D_I}) \xrightarrow{\delta} \bigoplus_{|I|=p+1} H^q(\Omega^\bullet_{D_I}) \xrightarrow{\delta} \bigoplus_{|I|=p+2} H^q(\Omega^\bullet_{D_I})$$

We separate the indices I into two groups: those that contain $i_0 = 1$ and those which do not. Note that if $1 \notin I$, then the natural map $H^q(\Omega^\bullet_{D_I}) \to H^q(\Omega^\bullet_{D_{\{1\}\cup I}})$ is an isomorphism by Theorem 3.5. It follows that a δ-cycle in the middle group only has components in the terms corresponding to I containing $i_0 = 1$. And these terms are all boundaries, *provided* $p > 0$. Thus after taking cohomology the only thing that remains is precisely $H^q(\Omega^\bullet_{D_1})$ and we are done. □

Definition 3.8. Let (R, I, γ) be a divided power ring killed by some non-zero integer. Let X be a separated, quasi-compact scheme over R. By Theorem 3.7, there is a unique (up to unique isomorphism) object $\mathscr{D}\Omega^\bullet_{X/R} \in D^+(X, R)$ equipped with isomorphisms $\phi_{\mathscr{U}} : \mathscr{D}\Omega^\bullet_{X/R} \to \Omega^\bullet_{\mathscr{U}}$ which are compatible with refinement. We call this object the *crystalline complex of X over R.* The hypercohomology groups of this complex are by definition the *crystalline cohomology of X over R:*

$$H^m_{crys}(X/R) = \mathbb{H}^m(X, \mathscr{D}\Omega^\bullet_{X/R}).$$

The assignment $X \mapsto \mathscr{D}\Omega^\bullet_{X/R}$ is functorial.

THEOREM 3.9. *Let X be a separated scheme, quasi-compact scheme over R. Consider a diagram*

$$\begin{tikzcd} X \arrow[r, hook, "closed"] \arrow[dr] & P \arrow[d, "smooth"] \\ & \operatorname{Spec} R \end{tikzcd}$$

Let X_i be an affine cover of P and let $U_i = X \cap X_i$ be the corresponding affine cover of X. Then the natural map

$$\Omega^\bullet_{X,Y} \to \Omega^\bullet_{\mathscr{U}}.$$

is a filtered quasi-isomorphism of R-modules.

PROOF. This is a local statement, so we can assume that X and Y are affine, in which case we are reduced to the case already treated. $\square$

In particular, $R\Gamma_{crys}(X/R)$ can be computed as $\Gamma(\Omega^\bullet_{\mathscr{U}})$ since

$$R\Gamma(X, j_{I*}\Omega^\bullet_{D_I}) = R\Gamma(X, Rj_{I*}\Omega^\bullet_{D_I}) = R\Gamma(U_I, \Omega^\bullet_{D_I}) = \Gamma(U_I, \Omega^\bullet_{D_I})$$

because U_I is affine and the components of $\Omega^\bullet_{D_I}$ are quasi-coherent sheaves on U_I.

5. Passing to the limit and finiteness

Up to now, we have always assumed that our base ring R was killed by some integer. In this section we discuss how to get rid of this hypothesis. Let (R, I, γ) be a divided power ring. Suppose that R is I-adically complete and separated, i.e.,

$$R = \varprojlim_n R/I^n.$$

Assume, furthermore, that $I \cap \mathbb{Z} \neq 0$ (that is, for any N, R/I^N is killed by some non-zero integer). We generally write $R_N = R/I^{N+1}$.

Our objective is to define crystalline cohomology of an R-scheme X over R. It turns out that the resulting object will only depend on the *formal completion of X over R*, a concept explained in Appendix B.

We assume that we are given a PD-ring (R, I, γ), where R ia complete and separated I-adic ring, and p is nilpotent in I^n for every n. We do *not* assume that R is Noetherian.

THEOREM 3.10. *The following are equivalent:*

- *$\overline{L}^*$ and $\overline{\alpha}$ can be lifted to a filtered quasi-isomorphism*

$$\alpha : L^* \to K^*$$

  with L^* filtered projective.
- *The induced map*

$$I \otimes_{\overline{R}} \overline{L}^* \to K_1^*$$

  is a filtered quasi-isomorphism. Here

$$\mathrm{Fil}^q(I \otimes_{\overline{R}} \overline{L}^n) = \sum_{a+b=q} \ldots$$

  and one checks that our map respects filtrations.

Proof. TBA. $\square$

Corollary 3.11. *Suppose $I \subset R$ is an ideal of R. Assume furthermore we have a projective system of filtered complexes K_n^* (bounded above) over $R_n = R/I^n$ (with induced filtration), such that the transition maps $\pi_n : K_n \to K_{n-1}$ are filtered surjective, and $\ker(\pi_n)$ is annihilated by I. Assume furthermore we have a filtered quasi-isomorphism $\alpha_1 : L_1^* \to K_1^*$ with L_1^* filtered projective, and that the obvious maps*

$$I^{n-1}/I^n \otimes_{R_1} L_1^* \to \ker(\pi_n)$$

are filtered quasi-isomorphisms. Then L_1^* lifts to a compatible system of filtered projective complexes L_n^* over R_n, and α_1^* to a compatible system of filtered quasi-isomorphisms $\alpha_n : L_n^* \to K_n^*$.

Theorem 3.12. *K_1^* can be represented by a filtered projective L^* such that*

- *$L^n = 0$ unless $0 \leq n \leq 2 \dim X$,*
- *all L^n are filtered direct summands in finite direct sums of copies of $R_1\{a\}$'s (such an L_1^* is called strictly perfect).*

Proof. TBA. $\square$

Theorem 3.13. *Assume $I^n = p \cdot R = 0$. Then K_n is filtered quasi-isomorphic to strictly perfect L_n^*.*

Proof. TBA. $\square$

Theorem 3.14. *Assume X is smooth over R_1. Then the construction above gives a complex, well-defined up to canonical quasi-isomorphism. Moreover it can be represented by a complex L^* all of whose terms are completions of filtered projective modules, and L^* is unique up to canonical filtered homotopy.*

If X/R_1 is proper, we can in addition assume that L^* is concentrated in degrees $[0, 2 \dim X]$, and that all L^m are filtered direct summands in finite direct sums of copies of $R\{a\}$'s. (L^* is strictly filtered perfect.)

CHAPTER 4

Around the Cartier isomorphism

Let p be a prime number. Suppose that X is a smooth quasi-projective scheme over $\mathbb{Z}_p$. Let $X_0 = X \otimes \mathbb{Z}/p$. The Frobenius map $F_0 : X_0 \to X_0$ (the definition of which is recalled in Section 1) induces a map:

$$F^* : \mathbb{H}^*(\widehat{\Omega}^\bullet_{X/\mathbb{Z}_p}) \longrightarrow \mathbb{H}^*(\widehat{\Omega}^\bullet_{X/\mathbb{Z}_p}).$$

The objective of this chapter is to study and exploit the existence of this extra "hidden" structure on the de Rham cohomology of X.

1. Frobeniuses

Let A be an algebra over $\mathbb{Z}/p$. Write $F_A : A \to A$ for the ring homomorphism $a \mapsto a^p$. Let X be a scheme over $\mathbb{Z}/p$. Let $F_X : X \to X$ be the absolute Frobenius of X defined by sending $x \in X(R)$ to $x \circ F_R^* : \operatorname{Spec} R \to X$. We call it the *absolute Frobenius of A*. If $X \to Y$ is a map of schemes over $\mathbb{Z}/p$, we have a commutative square

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Let $X^{(p)} = A \times_{Y,F_Y} Y$ (sometimes we will write this scheme as X'). The map F_X defines a map $F_{X/Y} : X^{(p)} \to X$ (*relative Frobenius*) fitting in the diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "F_{X/Y}"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Exercise 4.1. Let $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, $A = R[T]/(f)$, $f = \sum a_i T^i$. Then $X^p = \operatorname{Spec} A^{(p)}$, $A^{(p)} = R[T]/(f^{(p)})$, $f^{(p)} = \sum a_i^p T^i$ and $F^*_{X/Y}(T) = T^p$.

Exercise 4.2. Let $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[T]$. Compute $H^*(\Omega^\bullet_{X/\mathbb{F}_p})$. is an infinite-dimensional $\mathbb{F}_p$-vector space. More precisely, note that if $d(T^{pn}) = 0 \in \Omega^1_{\mathbb{F}_p[T]/\mathbb{F}_p}$ for every $n \geq 1$. Also, $T^{pn-1}dT \in \Omega^1_{\mathbb{F}_p[T]/\mathbb{F}_p}$ is a closed form that is not a boundary. Note that $P(T) \in \mathbb{F}_p[T]dT$ defines a cycle $dP = P'(T)dT = 0$ if and only if $P'(T) = 0$ if and only if $P(T) = Q(T^p)$ for some $Q \in \mathbb{F}_p[T]$. More generally, the image of $d : \mathbb{F}_p[T] \to \mathbb{F}_p[T]dT$ consists of forms $Q(T)\,dT$ with $Q(T) = \sum_{(n+1,p)=1} a_n T^n$, i.e., the image of d is $\bigoplus_{(n+1,p)=1} \mathbb{F}_p \cdot T^n dT$. Thus we see that $H^1(\Omega^\bullet_{\mathbb{F}_p[T]/\mathbb{F}_p}) = \bigoplus_k \mathbb{F}_p \cdot T^{pk-1}dT$. These facts generalise to any smooth $\mathbb{F}_p \to A$.

2. Cartier isomorphism

THEOREM 4.3. *Let $f : X \to Y$ be a map of schemes over $\mathbb{Z}/p$. There exists a unique homomorphism of $\mathscr{O}_{X^{(p)}}$-graded algebras*

$$C^{-1} = C^{-1}_{X/Y} : \bigoplus_i \Omega^i_{X^{(p)}/Y} \longrightarrow \bigoplus_i \mathscr{H}^i(F_{X/Y*}\Omega^\bullet_{X/Y})$$

such that

i) *For $i = 0$, it is induced by the map $F_{X/Y} : X \to X^{(p)}$.*
ii) *For $i = 1$, it is given by $\Omega^1_{X^{(p)}/Y} \to \mathscr{H}^1(\Omega^1_{X/Y})$, $1 \otimes db \mapsto b^{p-1}db$.*
iii) *If $X \to Y$ is smooth, then C^{-1} is an isomorphism.*

PROOF. First we construct the map. To define the map for $i = 1$, consider the map $B \to H^1(\Omega^\bullet_{B/A})$ that sends b to the class of $s^{p-1}ds$. Note that this class is closed. We check that it is a derivation

$$(st)^{p-1}d(st) = (st)^{p-1}sdt + (st)^{p-1}tds = s^pt^{p-1}dt + t^ps^{p-1}ds.$$

We also need to check additivity:

$$(s+t)^{p-1}d(s+t) - s^{p-1}ds - t^{p-1}dt = d\left(\frac{(s+t)^p - s^p - t^p}{p}\right).$$

Now suppose that $f : X \to Y$ is smooth. Note that the calculation in Exercise 4.2 proves the result when $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[T]$ and $Y = \operatorname{Spec} \mathbb{F}_p$.

In general we use *devissage* to reduce to this case. Since it is a local statement we can assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} R$, and that there is an etale $P \to B$ over R, with $P = R[X_1, \dots, X_d]$. Consider the diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[rr, bend left, "F_B"] & B^{(p)} \arrow[r] & B \\
P \arrow[u] \arrow[r] \arrow[rr, bend right, "F_P"'] & P^{(p)} \arrow[u] \arrow[r] & P \arrow[u]
\end{tikzcd}$$

The outer square is cartesian[1], so the right hand side square is also cartesian. It follows that the induced map

$$\Omega^\bullet_{P/R} \otimes_{P^{(p)}} B^{(p)} \to \Omega^\bullet_{B/R}$$

[1]Because $P \to B$ is etale. This implies that the induced map $B \otimes_{P,F_P} P \to B$ is etale. Since it is also radicial, it is an open immersion: see SGA 1 I 5.1. Since it is an homemomorphism to begin with, it must be an isomorphism.

is an an isomorphism of $B^{(p)}$-modules. Next, we consider the commutative diagram

$$\begin{tikzcd}
\Omega^i_{B^{(p)}/R} \arrow[rr, "C^{-1}_{B/R}"] & & H^i(\Omega^\bullet_{B/R}) \\
& & H^i(\Omega^\bullet_{P/R} \otimes_{P^{(p)}} B^{(p)}) \arrow[u, "\simeq"'] \\
\Omega^i_{P^{(p)}/R} \otimes_{P^{(p)}} B^{(p)} \arrow[uu, "\simeq"] \arrow[rr, "C^{-1}_{P/R}\otimes 1"] & & H^i(\Omega^\bullet_{P/R}) \otimes_{P^{(p)}} B^{(p)} \arrow[u, "\alpha"']
\end{tikzcd}$$

Note that the left vertical map is an isomorphism because $P^{(p)} \to B^{(p)}$ is the base change of $P \to B$ along $F_R : R \to R$ and therefore it is etale. Suppose that we have proven that $C^{-1}_{P/R}$ is an isomorphism. Then the lower horizontal map is an isomorphism. It follows that the map α is an isomorphism[2] and therefore $C^{-1}_{B/R}$ is an isomorphism. Hence we can assume that $B = P = R[X_1, \dots, X_d]$. Since the natural map $\bigwedge^i H^1(\Omega^\bullet_{P/R}) \to H^i(\Omega^\bullet_{P/R})$ is an isomorphism of $P^{(p)}$-modules in this case, we can assume that $d = 1$, i.e., $P = R[X]$. Finally, we write $R[X] = \mathbb{F}_p[X] \otimes_{\mathbb{F}_p} R$. Then $C^{-1}_{R[T]/R} = C^{-1}_{\mathbb{F}_p[T]/\mathbb{F}_p} \otimes R$ and we can assume $R = \mathbb{F}_p$, which is all that was left to do. $\square$

3. Crystalline Cartier isomorphism

In this section we will show that the Cartier isomorphism C^{-1} arises (under a liftability hypothesis) from a morphism in the level of complexes. We will define this morphism using crystalline cohomology.

Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ its ring of Witt vectors, X_s a smooth quasi-projective scheme over $W_s = W/p^{s+1}$, and $F_0 : X_0 \to X_0$ the absolute Frobenius endomorphism of $X_0 = X_s \otimes k$.

Let $X_s \subset P_s$ be any projective embedding. Let $D_m = D_{P_m}(X_m)$. Since $D_m = D_{P_m}(X_0)$, the canonical lift of Frobenius to P_m induces a map $F : D_m \to D_m$ (for any $m \leq s$).

$$F^* : \Omega^\bullet_{D_s} \to \Omega^\bullet_{D_s}$$

Theorem 4.4. *With the above notation, for each $r < p$ and any n such that $n + r \leq s$, there is a unique σ-semi-linear maps*

$$\phi_r : \mathrm{Fil}^r\, \Omega^\bullet_{D_n} \to \Omega^\bullet_{D_n}$$

such that $p^r \phi_r = F^*$.

The key observation needed to define ϕ_n is the following divisibility estimate. Let $\langle m \rangle$ be defined by $(p^{\langle m \rangle}) = (p)^{[m]}$ if $m \geq 0$ and zero otherwise.

Lemma 4.5. *With the above notation,*

$$F^*(\mathrm{Fil}^r\, \Omega^i_{D_m}) \subset p^{i+\langle r-i \rangle} \Omega^i_{D_m}.$$

---

[2]Use the spectral sequence arising from the identity $L\mathrm{id}(-) \otimes^L_{P^{(p)}} B^{(p)} = - \otimes^L_{P^{(p)}} B^{(p)}$,

$$\mathrm{Tor}^i_{P^{(p)}}(H^j(\Omega^\bullet_{P/R}), B^{(p)}) \implies H^{i+j}(\Omega^\bullet_{P/R} \otimes_{P^{(p)}} B^{(p)}).$$

and the fact that $P^{(p)} \to B^{(p)}$ is etale, hence flat.

Proof. Since $F^*(x) \equiv x^p \pmod p$, we see that $F^*(\Omega^i_{D_m}) \subset p^i\Omega^i_{D_m}$. Let J be the divided power ideal of D_m. If $x \in J$, then $F^*(x) = x^p + py$ for some $y \in D_m$ and

$$\begin{aligned} F^*(\gamma_e(x)) &= \gamma_e(F^*(x)) = \gamma_e(x^p + py) = \gamma_e(p!\gamma_p(x) + py) \\ &= ((p-1)!\gamma_p(x) + y)^e \gamma_e(p) \in (p)^{[e]}. \end{aligned}$$

Then $F^*(J^{[e]}) \subset (p)^{[e]} = (p^{\langle e \rangle})$ and the conclusion follows. □

Proof of Theorem 4.4. If $r < p$, then $\langle r-i \rangle = r-i$ for all $i \le r$ and in this case the above divisibility bound simplifies to

$$F^*(\mathrm{Fil}^r\,\Omega^\bullet_{D_m}) \subset p^r\Omega^\bullet_{D_m}.$$

In order to "divide by p^r", note that the terms of $\Omega^\bullet_{D_m}$ are flat modules over $\mathbb{Z}/p^{m+1}$ (Theorem 2.34). Setting $m = r+n$ and tensoring the injection $p^r : \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^{n+r+1}$ with $\Omega^\bullet_{D_{r+n}}$ we deduce that the map

$$p^r : \Omega^\bullet_{D_n} \to \Omega^\bullet_{D_{r+n}}.$$

is injective. Thus the map $F^* : \mathrm{Fil}^r\,\Omega^\bullet_{D_{r+n}} \to \Omega^\bullet_{D_{r+n}}$ factors through this injection and we can consider $p^{-r}F^* : \mathrm{Fil}^r\,\Omega^\bullet_{D_{r+n}} \to \Omega^\bullet_{D_n}$. Reducing modulo p^{n+1} gives a map $\phi_r : \mathrm{Fil}^r\,\Omega^\bullet_{D_n} \to \Omega^\bullet_{D_n}$ such that $p^r\phi_r = F^*$. Uniqueness is clear because $p^r : \Omega^\bullet_{D_n} \to \Omega^\bullet_{D_{r+n}}$ is injective. □

Suppose that $s \ge p-1$. Then we get a map lifting C^{-1} modulo quasi-isomorphisms.

Theorem 4.6. *Let X be a smooth W_s-scheme, $s \ge p-1$. There is a diagram of quasi-isomorphisms*

$$\begin{tikzcd} \bigoplus_{n<p} \Omega^n_{X_0'}[-n] \arrow[d, hook] \arrow[dr, dashed, "\widetilde{C}^{-1}"] & \\ \bigoplus_{n<p-1} \mathrm{gr}^n\,\Omega^\bullet_{X_0'} \oplus \mathrm{Fil}^{p-1}\,\Omega^\bullet_{X_0'} & \sigma_{<p}F_{0*}\Omega^\bullet_{X_0} \\ \bigoplus_{n<p-1} \mathrm{gr}^n\,\Omega^\bullet_{D_0'} \oplus \mathrm{Fil}^{p-1}\,\Omega^\bullet_{D_0'} \arrow[u, "q\text{-}iso"] \arrow[r, "\sum\phi_n"] & \sigma_{<p}F_*\Omega^\bullet_{D_0} \arrow[u, "q\text{-}iso"'] \end{tikzcd}$$

$$\widetilde{C}^{-1} : \bigoplus_{n<p} \Omega^n_{X_0'}[-n] \xrightarrow{\simeq} \sigma_{<p}F_{0*}\Omega^\bullet_{X_0}.$$

inducing C^{-1} in degrees $< p$ after taking cohomology.

Proof. This follows from Theorem 4.4 when $n = 0$. By construction $H^i(\widetilde{C}^{-1}) = C^{-1}$ for $0 \le i, n < p$. It follows from Cartier's Theorem 4.3 that it is a quasi-isomorphism. □

The following Exercise shows an example of what one gets by lifting C^{-1} to the derived category.

EXERCISE 4.7. Let X be a smooth and proper scheme over W, $X_0 = X \otimes k$. Show that if $\dim X < p$, then the Hodge-de Rham spectral sequence

$$E_1^{pq} = H^q(X_0, \Omega^p_{X_0}) \implies H^{p+q}_{dR}(X_0/k)$$

degenerates in E_1.

4. Fontaine-Laffaille modules

We keep the general notation of the previous section: k is a perfect field of characteristic $p > 0$, $W = W(k)$ is its ring of Witt vectors, X is a smooth projective scheme over W, and $F_0 : X_0 \to X_0$ is the Frobenius morphism of $X_0 = X \otimes k$.

4.1. Motivation. Suppose that the Hodge cohomology groups $H^q(X, \Omega^p_X)$ are torsion-free (hence free) modules over W (e.g. if X is an abelian scheme over W). Then the Hodge-de Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \implies H^{p+q}_{dR}(X/W)$$

degenerates at E_1. Indeed, since the Hodge groups are torsion-free, we can check the degeneration after inverting p. After doing this, one is left with the Hodge-de Rham spectral sequence for $X_{K_0} = X \otimes_W K_0$ over K_0, a field of characeristic zero, and its degeneration is a well-known result (see also Exercise **??**). In concrete terms, the degeneration implies that the natural maps

$$\mathbb{H}^n(X, \Omega_X^{\bullet \geq p}) \to H^n_{dR}(X/W)$$

are injective onto direct summands. Also, recall that, if $n < p$, we have defined maps

$$\phi_i : \mathbb{H}^m(X', \sigma_{\geq n}\Omega^\bullet_{X'}) \to H^m_{dR}(X/W)$$

using the crystalline Frobenius. Since $H^n_{dR}(X/W)$ is free, in particular compatible with all base changes, we see that if $m < p$, these maps induce

$$\sum \phi_i : \sigma^* \operatorname{gr} H^m_{dR}(X_0/k) \to H^m_{dR}(X_0/k)$$

which coincides with the Cartier isomorphism C^{-1} and therefore it is an isomorphism. Finally, note that by base change, this is equivalent to saying that

$$\sum_n \phi_n(H^m(X', \sigma_{\geq n}\Omega^\bullet_{X'})) = H^m_{dR}(X/W).$$

In this section we will see that all these facts hold (with some restrictions) for any smooth W-scheme X, even if the Hodge cohomology groups are not free over W.

4.2. Fontaine-Laffaille structure in crystalline cohomology. We begin by giving a name to the precise structure that we saw existed in the de Rham cohomology of our motivating example.

DEFINITION 4.8. Let $r \geq 1$. A *Fontaine-Laffaille module of length* r is a W-module of finite length M, endowed with a finite decreasing filtration by direct W-summands $\operatorname{Fil}^i M$, $i \in \mathbb{Z}$, such that $\operatorname{Fil}^0 M = M$, $\operatorname{Fil}^{r+1} M = 0$ and W-semi-linear maps $\phi_i : \operatorname{Fil}^i M \to M$ such that $\phi_i|_{\operatorname{Fil}^{i+1} M} = p\phi_{i+1}$ and $\sum_i \phi_i(\operatorname{Fil}^i M) = M$. Morphisms are W-linear maps that sends Fil^i to Fil^i and commute with the ϕ_i.

REMARK 4.9. By Nakayama's Lemma, the condition that $\sum_i \phi_i(\mathrm{Fil}^i M) = M$ is equivalent to the map

$$\phi = \sum \phi_i : \sigma^* \operatorname{gr} M/pM \to M/pM$$

being surjective. Since both sides are k-vector spaces of the same dimension, this map is surjective if and only if it is an isomorphism. Thus we see that this condition is a *linear-algebraic analog of the Cartier isomorphism.*

In practice, we will come across finite W-modules M, M_i together with maps $\phi_i : M^i \to M$, $\alpha_i : M^{i+1} \to M^i$ such that

i) The composition $M^{i+1} \to M^i \to M$ coincides with the map $M^{i+1} \to M$,
ii) The map $M^i \to M$ is an isomorphism for $i \leq 0$.
iii) The composition of ϕ^{i-1} with $M^i \to M^{i-1}$ is $p\phi^i$.

Consider the W-module $\widetilde{M}$ defined by the diagram

$$\begin{array}{ccccc} \bigoplus_{t=1}^r M^i & \longrightarrow & \bigoplus_{t=0}^r M^i & \longrightarrow & \widetilde{M} \longrightarrow 0 \\ (x_1, \ldots, x_r) & \longmapsto & (\alpha_0 x_1, \alpha_1 x_2 - p x_1, \ldots, -p x_r) & & \end{array}$$

Then condition iii) can be restated as saying that the ϕ_i induce a map $\overline{\phi} : \widetilde{M} \to M$.

LEMMA 4.10. *Let (M, M^i, ϕ^i) be as above. If $\sigma^* \widetilde{M} \to M$ is an isomorphism then (M, M^i, ϕ^i) is the underlying data of a Fontaine-Laffaille module.*

PROOF. We need to prove that the maps $M^i \to M^{i-1}$ are injections and direct summands. The key observation is that by definition $\mathrm{length}_W(\widetilde{M}) \geq \mathrm{length}_W(M^0) = \mathrm{length}_W(M)$. If there is equality then necessarily the left hand side map in the above sequence must be injective. That map's injectivity is equivalent to the injectivity of all the maps $M^i \to M^{i-1}$. Since this argument can be applied to $(M/p^n, M^i/p^n, \phi^i)$, we see that the maps $M^i/p^n \to M/p^n$ are injective for all i and n. It follows that they are direct summands (use the structure of modules over a PID). $\square$

The main theorem of this section follows. The proof is a reformulation of the proof in [4], where it is phrased in terms of sheaves on the crystalline site of X_0/W.

THEOREM 4.11. *Let X_s be a smooth, projective scheme over W_s. For any non-negative integers n, m, r such that $0 \leq m \leq r < p$ and $n + r \leq s$, the data*

$$\left(H^m_{dR}(X_n/W_n), (\mathbb{H}^m(X_n, \sigma_{\geq j}\Omega^\bullet_{X_n/W_n}))_{0\leq j\leq r}, (\phi_j)_{0\leq j\leq r}\right)$$

defines a Fontaine-Laffaille module of length r.

PROOF. We start by recalling a concrete realisation of the maps ϕ_r. Let $X \subset P$ be a projective embedding of X, and put $D_m = D_{P_m}(X_m)$. Since $D_m = D_{P_m}(X_0)$, the canonical lift of Frobenius to P induces compatible maps $F_m : D_m \to D_m$. We deduce a semi-linear map

$$F^* : \Omega^\bullet_{D_m} \to \Omega^\bullet_{D_m}.$$

Consider the short exact sequence of complexes

$$\begin{array}{c} 0 \longrightarrow \bigoplus_{t=1}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_n} \longrightarrow \bigoplus_{t=0}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_n} \longrightarrow \Lambda_n^r \longrightarrow 0 \\ (x_1, \dots, x_t) \longmapsto (x_1, x_2 - px_1, \dots, -px_r) \end{array}$$

Then there is an induced map $\overline{\phi} : \Lambda_n^r \to \Omega^\bullet_{D_n}$. We will prove that the induced map $\overline{\phi} : \mathbb{H}^m(X_0, \Lambda_n^t) \to \mathbb{H}^m(X_0, \Omega^\bullet_{D_n})$ is a semi-linear bijection and that the induced sequence

$$\bigoplus_{t=1}^{r} \mathbb{H}^m(X_0, \operatorname{Fil}^t \Omega^\bullet_{D_n}) \longrightarrow \bigoplus_{t=0}^{r} \mathbb{H}^m(X_0, \operatorname{Fil}^t \Omega^\bullet_{D_n}) \longrightarrow \mathbb{H}^m(X_0, \Lambda_n^r)$$

is short exact. Since $\Omega^{\bullet \geq t}_{X_n}$ is quasi-isomorphic to $\operatorname{Fil}^t \Omega^\bullet_{D_n}$, this (together with Lemma 4.10) will prove the theorem.

We do induction on m. The case $m = -1$ is obvious. Let us prove that $\overline{\phi}$ is bijective using induction on n. Consider the diagram

$$\begin{tikzcd} \bigoplus_{t=1}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_{n-1}} \arrow[r, hook] \arrow[d, hook, "p"] & \bigoplus_{t=0}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_{n-1}} \arrow[r, two heads] \arrow[d, hook, "p"] & \Lambda_{n-1}^r \arrow[d, "p"] \\ \bigoplus_{t=1}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_n} \arrow[r, hook] \arrow[d, two heads] & \bigoplus_{t=0}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_n} \arrow[r, two heads] \arrow[d, two heads] & \Lambda_n^r \arrow[d] \\ \bigoplus_{t=1}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_0} \arrow[r, hook] & \bigoplus_{t=0}^{r} \operatorname{Fil}^t \Omega^\bullet_{D_0} \arrow[r, two heads] & \Lambda_0^r \end{tikzcd}$$

The middle and left column are exact, as are all the rows. It follows by the long exact sequence of cohomology associated to this short exact sequence of complexes that the right hand column is also exact. Thus we have a semi-linear map of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \Lambda_{n-1}^r \arrow[r, "p"] \arrow[d, "\overline{\phi}"] & \Lambda_n^r \arrow[r] \arrow[d, "\overline{\phi}"] & \Lambda_0^r \arrow[r] \arrow[d, "\overline{\phi}"] & 0 \\ 0 \arrow[r] & \Omega^\bullet_{D_{n-1}} \arrow[r, "p"] & \Omega^\bullet_{D_n} \arrow[r] & \Omega^\bullet_{D_0} \arrow[r] & 0 \end{tikzcd}$$

It is easy to see that

$$\Lambda_0^r = \bigoplus_{n < r-1} \operatorname{gr}^n \Omega^\bullet_{D_0} \oplus \operatorname{Fil}^r \Omega^\bullet_{D_0}$$

Furthermore, $\overline{\phi}$ is simply the map that induces the Cartier isomorphism C^{-1}. Next, look at the induced map between long exact sequences of cohomology:

$$\begin{tikzcd} \mathbb{H}^m(X_0, \Lambda_{n-1}^r) \arrow[r] \arrow[d, "\simeq"] & \mathbb{H}^m(X_0, \Lambda_n^r) \arrow[r] \arrow[d, "\overline{\phi}"] & \mathbb{H}^m(X_0, \Lambda_0^r) \arrow[d, "\simeq"] \\ \mathbb{H}^m(X_0, \Omega^\bullet_{D_{n-1}}) \arrow[r] & \mathbb{H}^m(X_0, \Omega^\bullet_{D_n}) \arrow[r] & \mathbb{H}^m(X_0, \Omega^\bullet_{D_0}) \end{tikzcd}$$

The first map is an isomorphism by induction on n. The last map is also a bijection by the Cartier isomorphism[3]. It follows that the map $\mathbb{H}^m(X_0, \Lambda_n^r) \to \mathbb{H}^m(X_0, \Omega^\bullet_{D_n})$ is bijective for all n. By induction on m the sequence

$$0 \longrightarrow \bigoplus_{t=1}^r \mathbb{H}^m(X_0, \mathrm{Fil}^t\,\Omega^\bullet_{D_n}) \longrightarrow \bigoplus_{t=0}^r \mathbb{H}^m(X_0, \mathrm{Fil}^t\,\Omega^\bullet_{D_n}) \longrightarrow \mathbb{H}^m(X_0, \Lambda_n^r)$$

is exact. Then

$$\begin{aligned}\mathrm{length}_W\,\mathbb{H}^m(X_0, \Omega^\bullet_{D_n}) &= \mathrm{length}_W\,\mathbb{H}^m(X_0, \mathrm{Fil}^0\,\Omega^\bullet_{D_n})\\ &\leq \mathrm{length}_W\,\mathbb{H}^m(X_0, \Lambda_n^r)\end{aligned}$$

and the fact that $\overline{\phi}$ is a bijection implies that both lengths are equal and thus that the the above sequence is exact. □

When $n = s = \infty$, if we mod out by torsion, we obtain what is called a *strongly divisible lattice*.

A very particular implication of the above theorem is the following result, that does not seem to have a direct proof.

Exercise 4.12. Let X be a smooth and proper over W such that $\dim X < p$. Show that the Hodge-de Rham spectral sequence

$$E_1^{pq} = H^q(X_n, \Omega^p_{X_n/W_n}) \implies H^{p+q}_{dR}(X_n/W_n)$$

for X_n/W_n degenerates at E_1 for every n (including $n = \infty$).

5. Frobenius is an isogeny

One of the principal results of this section is that Frobenius admits an inverse up to some finite power of p.

We keep the general notation of the previous section: k is a perfect field of characteristic $p > 0$, $W = W(k)$ is its ring of Witt vectors, X_0 is a smooth quasi-projective k-scheme and $F_0 : X_0 \to X_0$ is the absolute Frobenius endomorphism.

Theorem 4.13. *Let X_0 be a smooth, quasi-projective scheme over k. Then the relative Frobenius $F_{X_0/k} : X_0 \to X_0'$ induces an isomorphism*

$$F^* : H^*_{crys}(X_0'/W) \otimes \mathbb{Q} \to H^*_{crys}(X_0/W) \otimes \mathbb{Q}$$

of $W[1/p]$-modules.

The statement is local. Thus we can assume we are in the following situation: X_0 admits a smooth formal scheme X over W and there is a map $F : X \to X$ lifting F_0 over $\sigma : W \to W$.

Note that the terms of the complex $\widehat{\Omega}^\bullet_{X/W}$ are p-adically complete, separated, and p-torsion-free. Since $F^*(\Omega^1_{X_0}) = 0$, it follows that $F^*(\widehat{\Omega}^i_{X'}) \subset p^i\widehat{\Omega}^i_X$ for all i. Thus we see that F^* induces a map of complexes

$$F^* : \widehat{\Omega}^\bullet_{X/W} \longrightarrow K \subset \widehat{\Omega}^\bullet_{X/W} \tag{4}$$

[3]This is because the natural map $H^m(X_0, \Omega^r_{X_0}[-r]) \to \mathbb{H}^m(X_0, \sigma_{\geq r}\Omega^\bullet_{X_0})$ is an isomorphism if $m \leq r$. To see this, consider the hypercohomology spectral sequence

$$H^j(X_0, \sigma_{\geq r}\Omega^i_{X_0}) \implies \mathbb{H}^{i+j}(X_0, \sigma_{\geq r}\Omega^\bullet_{X_0}).$$

Since $m \leq r$, the group $\mathbb{H}^m(X_0, \sigma_{\geq r}\Omega^\bullet_{X_0})$ can only be non-zero only if $m = r$ and, if this is the case, the only non-zero term that can contribute from the left hand side is when $i = r$, $j = m - r$.

where $K^i = \{x \in p^i\widehat{\Omega}^i_{X/W} : dx \in p^{i+1}\widehat{\Omega}^{i+1}_{X/W}\}$. We will prove that (4) is a quasi-isomorphism. However, we need a more general statement for the proof.

In order to formulate this statement, we introduce notation to specify p-divisibility of the different terms of a complex. Given a function $\epsilon : \mathbb{Z} \to \mathbb{N}$ and a complex of sheaves K on X we define a subcomplex K_ϵ by

$$K^i_\epsilon = \{x \in p^{\epsilon(i)}K^i \mid dx \in p^{\epsilon(i+1)}K^{i+1}\} \subset p^{\epsilon(i)}K^i \subset K^i.$$

Note that $K^i_\epsilon = p^{\epsilon(i)}K^i$ if ϵ is decreasing.

Let $\epsilon : \mathbb{Z} \to \mathbb{N}$ be a function and set $\eta(i) = \epsilon(i) + i$. Then F^* induces a linear map

$$\Psi_\epsilon : \widehat{\Omega}^\bullet_{X',\epsilon} \to F_{X/W*}\widehat{\Omega}^\bullet_{X,\eta}.$$

We will say that ϵ is a *gauge* if $\epsilon(i) \geq \epsilon(i+1) \geq \epsilon(i) - 1$ for all $i \geq 0$. We can now formulate the more general statement.

Theorem 4.14. *Let ϵ be a gauge. Then the map Ψ_ϵ is a quasi-isomorphism.*

The proof has two parts. First we will prove the Theorem for gauges ϵ of a special form. Secondly, we will reduce the case of an arbitrary ϵ to this special case. To deal with the special case, let us say that ϵ is *steep* if $\epsilon(i+1) = \epsilon(i) - 1$ for all $i \geq 0$.

Lemma 4.15. *Let ϵ be a steep gauge. Then Ψ_ϵ is a quasi-isomorphism.*

Proof. Since the terms of the complex $\widehat{\Omega}^\bullet_X$ are p-adically complete and separated and p-torsion-free, it suffices to prove that $\Psi_\epsilon \otimes \mathbb{Z}/p$ is a quasi-isomorphism[4]. For any $j \geq 0$, let $e = \epsilon(j)$ and contemplate the following commutative diagram.

$$\begin{tikzcd}
p^e\widehat{\Omega}^j_{X'}/p^{e+1}\widehat{\Omega}^j_{X'} \arrow[rrr] \arrow[dr, "p^{-e}", "\simeq"'] \arrow[ddd, "F"'] & & & p^{e-1}\widehat{\Omega}^{j+1}_{X'}/p^e\widehat{\Omega}^{j+1}_{X'} \arrow[dl, "p^{-(e-1)}"', "\simeq"] \arrow[ddd, "F"] \\
& \Omega^j_{X'_0} \arrow[r, "0"] \arrow[d, "“p^{-j}F”"'] & \Omega^{j+1}_{X'_0} \arrow[d, "“p^{-(j+1)}F”"] & \\
& F_{0*}\Omega^j_{X_0} \arrow[r, "d"] & F_{0*}\Omega^{j+1}_{X_0} & \\
F_*(p^{e+j}\widehat{\Omega}^j_X/p^{e+j+1}\widehat{\Omega}^j_X) \arrow[rrr] \arrow[ur, "\simeq", "p^{-(e+j)}"'] & & & F_*(p^{e+j}\widehat{\Omega}^{j+1}_X/p^{e+j+1}\widehat{\Omega}^{j+1}_X) \arrow[ul, "\simeq"', "p^{-(e+j)}"]
\end{tikzcd}$$

[4]Considering the cone of Ψ_ϵ, we see that we must show that if C is a p-adically complete and separated, p-torsion-free complex such that C/pC is exact, then C is exact. Let $[x_0] \in H^i(C)$. Then $x_0 = dy_0 + px_1$ for some $y_0 \in C^{i-1}$, $x_1 \in C^i$. Then $pdx_1 = dx_0 - d^2y_0 = 0$ so $dx_1 = 0$. Repeating the argument, we find $x_1 = dy_1 + px_2$, so that $x_0 = d(y_0 + y_1p) + p^2x_2$. Iterating this we find $y_0, y_1, \ldots \in C^{i-1}$, $x_1, x_2, \ldots \in C^i$ such that $x_0 = d(y_0 + y_1p + y_2p^2 + \cdots + y_rp^r) + p^{r+1}x_{r+1}$. Letting $r \to \infty$ we obtain $x_0 = d\sum y_ip^i$ and we are done.

Taking cohomology sheaves we obtain a commutative diagram of the form

$$\begin{tikzcd}
\mathscr{H}^j(\widehat{\Omega}^\bullet_{X',\epsilon}\otimes\mathbb{Z}/p) \arrow[r, "\Psi_\epsilon\otimes\mathbb{Z}/p"] \arrow[d, "\simeq"] & \mathscr{H}^j(F_{X/W*}\widehat{\Omega}^\bullet_{X,\eta}\otimes\mathbb{Z}/p) \arrow[d, "\simeq"] \\
\Omega^j_{X_0'} \arrow[r, "C^{-1}", "\simeq"'] & \mathscr{H}^j(F_{X_0/k*}\Omega^\bullet_{X_0})
\end{tikzcd}$$

Thus $\Psi_\epsilon\otimes\mathbb{Z}/p$ is a quasi-isomorphism and we are done. □

Next, we will reduce the case of an arbitrary gauge ϵ to the previously treated case. We say that ϵ is *almost equal* to ϵ' if $\epsilon(i)=\epsilon'(i)$ for all i except at most for one value i_0 where $\epsilon(i_0)=\epsilon'(i)\pm 1$. Suppose that ϵ is almost equal to ϵ', $\epsilon'(j)=\epsilon(j)+1$ (in particular, $\epsilon(j-1)=\epsilon'(j-1)=\epsilon'(j)$).

LEMMA 4.16. *Let ϵ and ϵ' be two almost equal gauges. Then Ψ_ϵ is a quasi-isomorphism if and only if $\Psi_{\epsilon'}$ is one.*

PROOF. Note that $\widehat{\Omega}^\bullet_{X',\epsilon}/\widehat{\Omega}^\bullet_{X',\epsilon'}$ is only non-zero in degree j. Similarly, $\widehat{\Omega}^\bullet_{X,\eta}/\widehat{\Omega}^\bullet_{X,\eta'}$ is zero except in degrees $j-1$ and j. We claim that $\mathscr{H}^{j-1}(F_{X/W*}\widehat{\Omega}^\bullet_{X,\eta}/\widehat{\Omega}^\bullet_{X,\eta'})=0$ and that there is a commutative diagram

$$\begin{tikzcd}
\mathscr{H}^j(\widehat{\Omega}^\bullet_{X',\epsilon}/\widehat{\Omega}^\bullet_{X',\epsilon'}) \arrow[r, "\Psi_\epsilon"] \arrow[d, "\text{“}p^{-\epsilon(j)}\text{”}", "\simeq"'] & \mathscr{H}^j(F_{X/W*}\widehat{\Omega}^\bullet_{X,\eta}/\widehat{\Omega}^\bullet_{X,\eta'}) \arrow[d, "\text{“}p^{-\eta(j)}\text{”}", "\simeq"'] \\
\Omega^j_{X_0'} \arrow[r, "C^{-1}", "\simeq"'] & \mathscr{H}^j(F_{X_0/k*}\Omega^\bullet_{X_0})
\end{tikzcd}$$

To see this, we just write everything out in terms of the definitions. Let $n=\eta(j-1)=\eta'(j-1)=\eta'(j)-1=\eta(j)$. Then

$$\begin{aligned}
\widehat{\Omega}^{j-1}_{X,\eta}/\widehat{\Omega}^{j-1}_{X,\eta'} &= \frac{\{\omega\in p^n\widehat{\Omega}^{j-1}_X\}}{\{\omega\in p^n\widehat{\Omega}^{j-1}_X : d\omega\in p^{n+1}\widehat{\Omega}^j_X\}} \\
&\xrightarrow[\simeq]{p^{-n}} \frac{\{\omega\in\widehat{\Omega}^{j-1}_X\}}{\{d\omega\in p\widehat{\Omega}^j_X\}} \xrightarrow[\simeq]{d} \operatorname{im}(d:\Omega^{j-1}_{X_0}\to\Omega^j_{X_0}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{\Omega}^{j}_{X,\eta}/\widehat{\Omega}^{j}_{X,\eta'} &= \frac{\{\omega\in p^n\widehat{\Omega}^{j}_X : d\omega\in p^{n+1}\widehat{\Omega}^{j+1}_X\}}{\{\omega\in p^{n+1}\widehat{\Omega}^j_X\}} \\
&\xrightarrow[\simeq]{p^{-n}} \frac{\{\omega\in\widehat{\Omega}^{j}_X : d\omega\in p\widehat{\Omega}^{j+1}_X\}}{\{\omega\in p\widehat{\Omega}^j_X\}} \xrightarrow[\simeq]{} \ker(d:\Omega^{j}_{X_0}\to\Omega^{j+1}_{X_0}).
\end{aligned}$$

Thus we see that if $B^j_{X_0}=\operatorname{im}(d:\Omega^{j-1}_{X_0}\to\Omega^j_{X_0})$ and $Z^j_{X_0}=\ker(d:\Omega^j_{X_0}\to\Omega^{j+1}_{X_0})$, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{H}^{j-1}(\widehat{\Omega}^{j-1}_{X,\eta}/\widehat{\Omega}^{j-1}_{X,\eta'}) \arrow[r, hook] \arrow[d, dashed, "\simeq"] & \widehat{\Omega}^{j-1}_{X,\eta}/\widehat{\Omega}^{j-1}_{X,\eta'} \arrow[r, "d"] \arrow[d, "p^{-n}d", "\simeq"'] & \widehat{\Omega}^{j}_{X,\eta}/\widehat{\Omega}^{j}_{X,\eta'} \arrow[r, two heads] \arrow[d, "p^{-n}", "\simeq"'] & \mathscr{H}^j(\widehat{\Omega}^\bullet_{X,\eta}/\widehat{\Omega}^\bullet_{X,\eta'}) \arrow[d, dashed, "\simeq"] \\
0 \arrow[r] & B^j_{X_0} \arrow[r, hook] & Z^j_{X_0} \arrow[r, two heads] & \mathscr{H}^j(\Omega^\bullet_{X_0})
\end{tikzcd}$$

The claim follows at once. The long exact sequence in cohomology then implies that Ψ_ϵ is a quasi-isomorphism if and only if $\Psi_{\epsilon'}$ is one. $\square$

Proof of Theorem 4.14. Since the value of ϵ at k is irrelevant if $\Omega^k_X = 0$, we can assume that $\epsilon(k) = 0$ for $k \gg 0$. Then it is clear that we can find a sequence of gauges $\epsilon = \epsilon_0, \dots, \epsilon_N = \epsilon'$ such that ϵ' is steep (draw a picture!). The repeated application of Lemma 4.16 and Lemma 4.15 finish the proof. $\square$

Proof of Theorem 4.13. Since this is a local statement, we can assume that X_0 is affine. Then it admits a smooth formal lift X over W. By smoothness we can also lift F_0 to $F : X \to X$. Theorem 4.14 with $\epsilon = 0$ shows that it is enough to prove that the inclusion $\widehat{\Omega}^\bullet_{X/W,\eta} \subset \widehat{\Omega}^\bullet_{X/W}$ becomes an isomorphism after inverting p, which is obvious. $\square$

6. Berthelot-Ogus isomorphism

We have already discussed that if X is a smooth, projective scheme over W and $X_0 = X \otimes_W k$, then there is a canonical isomorphism

$$H^*_{crys}(X_0/W) \simeq H^*_{dR}(X/W).$$

We are interested in generalising this statement to arbitrary finite extensions of $\mathbb{Z}_p$.

Let us put ourselves in the following situation: k is a perfect field of characteristic $p > 0$, $W = W(k)$ is its ring of Witt vectors, V is a finite extension of W with residue field k, $K = V[1/p]$ is the ring of fractions, X is a smooth projective V-scheme, and $X_0 = X \otimes k$ is its closed fibre.

The difficulty lies in the fact that the kernel of the projection $V \to k$ does not, in general, have divided powers. If $\pi \in V$ is a uniformiser, the kernel is precisely $\pi V \subset V$ and this ideal has divided powers if and only if $e < p$, where e is the unique integer such that $\pi^e V = pV$. If this is the case, then we can use $(V, \pi V, \mathrm{std})$ as our base divided power ring and deduce a canonical isomorphism

$$H^*_{crys}(X_0/W) \otimes_W V \simeq H^*_{crys}(X_0/V) \simeq H^*_{dR}(X/V)$$

(where the isomorphism on the left results from the Base Change Theorem **??** and the flatness of V over W). In this section we will prove that such an isomorphism exists for any e if we allow ourselves to invert p.

Theorem 4.17. *Let V be a finite extension of $\mathbb{Z}_p$ with residue field k. Let $W = W(k)$, $K = V[1/p]$. Let X be a smooth quasi-projective V-scheme. Then there are functorial isomorphisms*

$$R\Gamma(X_0, \mathscr{D}\Omega^\bullet_{X_0/W}) \otimes^L_W K \simeq R\Gamma(X, \Omega^\bullet_{X/V}) \otimes^L_V K.$$

In particular, if X is projective over V, one has isomorphisms of K-vector spaces

$$H^*_{crys}(X_0/W) \otimes_W K \simeq H^*_{dR}(X/V) \otimes_V K.$$

Proof. By the Base Change Theorem **??** we have a natural isomorphism

$$R\Gamma_{crys}(X_0/W) \otimes^L_W V \xrightarrow{\simeq} R\Gamma_{crys}(X_0 \otimes_k V/pV/V).$$

We want to compare the crystalline cohomology of $X_0 \otimes_k V/pV$ over V and that of $X \otimes_V V/pV$ over V. We know that there is a natural isomorphism

$$R\Gamma_{crys}(X \otimes_V V/pV/V) = R\Gamma_{dR}(X/V).$$

Note that some power of $F_V : V/pV \to V/pV$ descends to $V/\pi V = k$. Indeed, if $n \geq e$, $\pi^e V = pV$, then $F_V^e(\pi) \equiv 0 \pmod p$. It follows that if $n \geq e$, we have canonical identifications

$$(X \otimes_V V/pV) \otimes_{V/pV, F_V^n} V/pV \simeq (X \otimes k) \otimes_{k, F_V^n} V/pV.$$

It follows that we have a diagram

$$\begin{tikzcd}
R\Gamma((X \otimes_V V/pV)^{(n)}/V) \otimes_V^L \mathbb{Q} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & R\Gamma(X \otimes_V V/pV/V) \otimes_V^L \mathbb{Q} \arrow[d, dashed, "\alpha_X"] \\
R\Gamma((X_0 \otimes_k V/pV)^{(n)}/V) \otimes_V^L \mathbb{Q} \arrow[r, "\simeq"] & R\Gamma(X_0 \otimes_k V/pV/V) \otimes_V^L \mathbb{Q}
\end{tikzcd}$$

It is easy to see that the dashed map α is independent of n and that α_X depends functorially on X. The statement about cohomology groups follows by flatness of K over W and V. $\square$

APPENDIX A

Smoothness and differential forms

1. Kähler differentials

Let $R \to B$ be a ring homomorphism. An R-derivation of B into a B-module M is an additive map $D : B \to M$ such that $D(b_1 b_2) = b_1 D(b_2) + b_2 D(b_1)$ for all $b_1, b_2 \in B$ and such that $D(r) = 0$ if $r \in R$. The *module of Kähler differentials* of B over R is an B-module equipped with a R-derivation $d : B \to \Omega^1_{B/R}$ which is universal in the following sense: for each R-derivation $D : B \to M$, there is a unique B-linear homomorphism $u : \Omega^1_{B/R} \to M$ such that $D = ud$.

Existence can be checked easily in the following way. Suppose first that $B = R[T_\alpha]_\alpha$. Let Ω^1 be the the free B-module with basis dT_α and define $d : B \to \Omega^1$ by the formula

$$d(P) = \sum_\alpha \frac{\partial P}{\partial T_\alpha} dT_\alpha.$$

It is very easy to check that (Ω^1, d) satisfies the universal property. In general, write $B = B_1/K$ with $B_1 = R[T_\alpha]_\alpha$ and some ideal K. Then one checks that

$$\Omega^1_{B/R} = \Omega^1_{B_1/R}/(K\Omega^1_{B_1/R} + B_1 dK).$$

(with the induced d) satisfies the universal property.

Exercise A.1. In this exercise we give a more canonical construction of $\Omega^1_{B/R}$.

Exercise A.2. Let $R \to B_1$, $R \to B_2$ be ring homomorphisms. Then the map

$$\Omega^1_{B_1 \otimes_R B_2/R} \to \Omega^1_{B_1/R} \boxtimes \Omega^1_{B_2/R}$$

induced by $d(b_1 \otimes b_2) = b_2 db_1 \otimes b_1 db_2$ is an isomorphism. Suppose that S is a multiplicative set of B. Then there is a canonical isomorphism

$$\Omega^1_{S^{-1}B/R} \to S^{-1}\Omega^1_{B/R}$$

sending $d(bs^{-1}) = s^{-2}(sdb - bds)$. Finally, suppose that $R \to R'$ is any ring homomorphism. Put $B' = B \otimes_R R'$. Then the natural map

$$\Omega^1_{B/R} \otimes_B B' \to \Omega^1_{B'/R'}$$

is an isomorphism.

Exercise A.3. Let $R \to A \to B$ be ring homomorphisms such that $B = A/J$. Then the map $d : J \to \Omega^1_{A/R}$ induces an exact sequence

$$J/J^2 \to \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to 0.$$

Exercise A.4. Let $R \to A \to B$ be ring homomorphisms. Then there is an exact sequence

$$\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0.$$

2. Square-zero extensions

Let $R \to B$ be a ring homomorphism. Let J be a module over B.

DEFINITION A.5. A *square zero extension of B by J over R* is a ring homomorphism $B' \to B$ of algebras over R, surjective, with kernel of square zero, isomorphic to J as module over B. We note any such extension as

$$\begin{tikzcd} J \arrow[r, hook] & B' \arrow[r, two heads] & B \\ & R \arrow[u] \arrow[ur] & \end{tikzcd}$$

Two extensions $[J \to B'_1 \to B]$ and $[J \to B'_2 \to B]$ are isomorphic if there exists a diagram

$$\begin{tikzcd} & B'_1 \arrow[dd, dashed] \arrow[dr, two heads] & \\ J \arrow[ur, hook] \arrow[dr, hook] & & B \\ & B'_2 \arrow[ur, two heads] & \end{tikzcd}$$

Note that the dashed arrow is automatically an isomorphism. We let $\mathrm{Ext}_R(B, J)$ be the set of isomorphism classes of square-zero extensions of B by J over R.

EXERCISE A.6. Let J be a module over B. Then $B' = B \oplus J$ can be made into a ring with multiplication $(b, x) \cdot (b', x') = (bb', bx' + xb)$. With this structure $J \to B' \to B$ is a square-zero extension. Moreover, $J \mapsto B \oplus J$ is functorial and given an extension of B-modules $0 \to J \to E \to M \to 0$, it induces a square-zero extension $0 \to J \to B \oplus E \to B \oplus M \to 0$.

We define a group structure on $\mathrm{Ext}_R(B, J)$ as follows. Suppose given two square-zero extensions

$$\begin{tikzcd} J \arrow[r, hook] & B'_1 \arrow[r, two heads] & B \end{tikzcd}$$

$$\begin{tikzcd} J \arrow[r, hook] & B'_2 \arrow[r, two heads] & B \end{tikzcd}$$

Define $B'_3 = J \oplus_{J \oplus J} (B'_1 \times_B B'_2)$. This is naturally a square-zero extension of B by J over R and it defines an abelian group structure on $\mathrm{Ext}_R(B, J)$. The neutral element of $\mathrm{Ext}_R(B, J)$ is the class of the square-zero extension

$$\begin{tikzcd} J \arrow[r, hook] & B \oplus J \arrow[r, two heads] & B \end{tikzcd}$$

Where the multiplication on $B \oplus J$ is given by $(b_1, x_1) \cdot (b_2, x_2) = (b_1 b_2, b_1 x_2 + b_2 x_1)$.

THEOREM A.7. *Let $R \to A \to B$ be ring homomorphisms. Then there is an exact sequence*

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_A(B,J) \arrow[r] & \mathrm{Der}_R(B,J) \arrow[r] & \mathrm{Der}_R(A,J) \arrow[dll, "\delta"] \\
& \mathrm{Ext}_A(B,J) \arrow[r] & \mathrm{Ext}_R(B,J) \arrow[r] & \mathrm{Ext}_R(A,J)
\end{tikzcd}
$$

PROOF. Long, but routine. □

3. Smoothness

DEFINITION A.8. A ring homomorphism $R \to B$ is *formally smooth* if for each homomorphism $R \to C$ and ideal $I \subset C$ such that $I^2 = 0$ and every homomorphism $\overline{\phi} : B \to C/I$ of R-algebras, there is a map $\phi : B \to C$ of R-algebras lifting $\overline{\phi}$.

$$
\begin{tikzcd}
B \arrow[r, "\overline{\phi}"] \arrow[dr, dashed, "\phi"] & C/I \\
R \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}
$$

$R \to B$ is *smooth* if is formally smooth and of finite presentation. If the map ϕ is unique for all $\overline{\phi}$, we say that $R \to B$ is etale.

REMARK A.9. The set of liftings ϕ of $\overline{\phi}$ is a torsor under the set of derivations $\mathrm{Der}_R(B, I)$. That is, if ϕ_1 is a lifting of $\overline{\phi}$, then ϕ_2 is another such lifting if and only if $\phi_1 - \phi_2 \in \mathrm{Der}_R(B, I)$.

EXERCISE A.10. Show that $R[T_1, \dots, T_N]$ is a smooth algebra over R.

EXERCISE A.11. If $R \to B$ is smooth and $B \to C$ is smooth then $R \to C$ is smooth. If $R \to B$ is smooth and $f \in B$, then $B \to B_f$ is smooth. Let $R \to S$ be a ring homomorphism. If $R \to B$ is smooth, then $S \to S \otimes_R B$ is smooth. The same results hold replacing 'smooth' everywhere for 'etale'.

EXERCISE A.12. Let $R \to B$ be a ring homomorphism. Let $f_1, \dots, f_r \in B$ be such that $(f_1, \dots, f_r) = B$ and such that $R \to B_{f_i}$ is finitely presented. Then $R \to B$ is finitely presented.

THEOREM A.13. *Let $R \to B$ be a ring homomorphism. Suppose that there exist $f_1, \dots, f_r \in B$ such that $(f_1, \dots, f_r) = B$ and $R \to B_{f_i}$ is smooth. Then $R \to B$ is smooth.*

PROOF. Let $\overline{\phi} : B \to C/I$ be as in Definition A.8. Then there exists $\phi_i : B_{f_i} \to C_{f_i}$ lifting $\overline{\phi} : B_{f_i} \to C_{f_i}/IC_{f_i}$. Then $\phi_i - \phi_j \in Der(B, IC_{f_i f_j})$. This gives a 1-cocycle in $H^1(\mathrm{Spec}(C/I), \mathrm{Der}(B, I))$. Since $\mathrm{Spec}(C/I)$ is affine, the cohomology group is zero, so that there is a $\phi : B \to C$ lifting $\overline{\phi}$. □

DEFINITION A.14. A ring homomorphism $R \to B$ is *standard smooth* if it has a presentation

$$B = R[T_1, \dots, T_n]/(f_1, \dots, f_r)$$

such that $\det(\partial f_i / T_j)_{i,j=1,\dots,r} \in B^\times$.

LEMMA A.15. *A standard smooth algebra $R \to B$ is smooth. It is etale if and only if $n = r$.*

PROOF. Let $\overline{\varphi} : B \to C/I$ be a map of algebras over R, where $I^2 = 0$. Let $c_1, \dots, c_n \in C$ such that $\varphi(T_i) \equiv c_i \pmod{I}$. We want to find $h_1, \dots, h_n \in I$ such that $0 = f_i(c_1 + h_1, \dots, c_n + h_n) \in C$. Since $I^2 = 0$, we can look at the Taylor expansion of the f_i:

$$0 \stackrel{?}{=} f_i(c_1 + h_1, \dots, c_n + h_n) = f_i(c_1, \dots, c_n) + \sum_{j=1}^{n} \frac{\partial f_i}{\partial T_j}(c_1, \dots, c_n) h_j.$$

By the determinant condition we can find the required h_i. There is only one choice if $n = r$. $\square$

REMARK A.16. Let $R \to B$ be a standard smooth homomorphism. Then $\Omega^1_{B/R}$ is free over B. In fact, a basis is given by $dT_{r+1}, \dots, dT_n$.

THEOREM A.17. *Let $R \to B$ be a standard smooth homomorphism. Then $R \to B$ is flat.*

PROOF. How? A simple proof would be nice. $\square$

4. Criteria for smoothness

LEMMA A.18. *Let $R \to B$ be a ring homomorphism. Then $R \to B$ is formally smooth if and only if $\mathrm{Ext}_R(B, J) = 0$ for every module J over B.*

PROOF. Suppose that $R \to B$ is smooth. Then every square-zero extension of B by J over R admits a splitting so that $\mathrm{Ext}_R(B, J) = 0$. Conversely, suppose that $\mathrm{Ext}_R(B, J) = 0$ for every J. Let $B \to C/I$ be as in Definition A.8. Let $E = B \times_{C/I} C$ be the fibre product of B and C over C/I. Then $I \to E \to B$ is a square-zero extension of B by I over R. Hence it admits a splitting $B \to E$. Composing with the projection $E \to C$ gives the desired lift of $B \to C/I$. $\square$

THEOREM A.19. *Let $R \to A \to B$ be ring homomorphisms such that $B = A/J$. If $R \to B$ is smooth, the sequence*

$$0 \to J/J^2 \to \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to 0$$

is split exact. If the sequence is split exact and $R \to A$ is smooth, then $R \to B$ is smooth.

PROOF. Suppose first that $R \to B$ is smooth. We can fill the diagram:

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[dr, dashed, "\phi"] & A/J \\
R \arrow[u] \arrow[r] & A/J^2 \arrow[u, two heads, "\pi"']
\end{tikzcd}$$

We obtain a splitting of the sequence

$$0 \to J/J^2 \to A/J^2 \to A/J \to 0$$

In particular a section of the inclusion $J/J^2 \to A/J^2$ is given by $\psi = 1 - \phi\pi$. It is easy to check that it is a R-derivation. Hence it induces a A-linear homomorphism $\Omega^1_{A/R} \to J/J^2$ splitting the natural map and we are done.

Next, suppose that the sequence is split exact and that $R \to A$ is smooth. Let $\overline{\phi} : B \to C/I$ be as in Definition A.8. Then by smoothness of $R \to A$, we can find a map $\psi : A \to C$ lifting $A \to B \to C/I$. We will modify ψ by adding a derivation $D \in \mathrm{Der}_R(A, I)$ so that $\psi = D$ on J. This is clearly enough. Since $\psi(J) \subset I$ and $I^2 = 0$, we get a map $\psi' : J/J^2 \to I$. As the sequence is split exact we can extend this to a map $\psi'' : \Omega^1_{A/R} \to I$. Then $D = \psi'' d : A \to I$ is the derivation we are looking for. Indeed, if $x \in J$, $D(x) = \psi'' d(x) = \psi'(x)$ so we are done. $\square$

Corollary A.20. *Let $R \to B$ be a smooth homomorphism. Then there is a partition of unity $(f_1, \ldots, f_r)$ in B such that $R \to B_{f_i}$ is a standard smooth homomorphism.*

Proof. Write $B = A/J$ with $A = R[T_1, \ldots, T_n]$ and $J \subset A$ a finitely generated ideal. Let $x \in \operatorname{Spec} B$, $k = k(x)$. By Theorem A.19 the sequence

$$0 \to J/J^2 \otimes k \to \Omega^1_{A/R} \otimes k \to \Omega^1_{B/R} \otimes k \to 0$$

is exact. Shrinking if necessary and using Nakayama's Lemma, we see that there exists a generating set $s_1, \ldots, s_r \in J$ such that $ds_1(x), \ldots, ds_r(x) \in \Omega^1_{A/R} \otimes k$ are linearly independent. This is what we need. $\square$

Corollary A.21. *Let R be a ring, $I \subset R$ a nilpotent ideal. Let $R/I \to B$ be a smooth morphism. Then there exists a smooth $R \to C$ such that $B = C \otimes_R R/I$.*

Proof. Later. $\square$

Theorem A.22. *Let $R \to B$ be a smooth homomorphism. The map $\varphi : \mathrm{Ext}_R(B, J) \to \mathrm{Ext}^1_B(\Omega^1_{B/R}, J)$ sending $0 \to J \to B' \to B \to 0$ to $0 \to J \to \Omega^1_{B'/R} \otimes_{B'} B \to \Omega^1_{B/R} \to 0$ is an isomorphism of abelian groups.*

Proof. TBA. $\square$

Corollary A.23. *Let $R \to B$ be a smooth homomorphism. Then $\Omega^1_{B/R}$ is finite projective.*

Proof. This follows from Lemma A.18 and Theorem A.22, because $\Omega^1_{B/R}$ is of finite type and it is projective because $\mathrm{Ext}^1_B(\Omega^1_{B/R}, J) = \mathrm{Ext}_R(B, J) = 0$ for every B-module J. $\square$

Theorem A.24. *Let $R \to A \to B$ be ring homomorphisms. Suppose that $A \to B$ is smooth. Then the sequence*

$$0 \to \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

is split exact. If the sequence is split exact and $R \to B$ is smooth, then $A \to B$ is smooth.

Proof. Suppose first that the sequence is split exact and that $R \to B$ is smooth. Let J be a B-module. Looking at the exact sequence of Ext, we see that since B is smooth over R, $\mathrm{Ext}_R(B, J) = 0$. Since the sequence is split exact, the map $\mathrm{Der}_R(B, J) \to \mathrm{Der}_R(A, J)$ is surjective, so that $\mathrm{Ext}_A(B, J) = 0$ and we are done.

Now assume that B is smooth over A. To check that the sequence is split exact it is enough to check that the sequence remains exact after applying the functor $\mathrm{Hom}_B(-, I)$ for any B-module I. We use the exact sequence of Theorem A.7. Since $\mathrm{Ext}_A(B, I) = 0$ by Lemma A.18, we are done. $\square$

5. Implicit function theorem

THEOREM A.25. *Let $R \to A \to B$ be ring homomorphisms such that $R \to A$ and $R \to B$ are smooth and $B = A/J$. Then there exists a partition of unity $(f_1, \ldots, f_d)$ in A such that*

$$\begin{tikzcd} A_{f_i} \arrow[rr, two heads] & & B_{f_i} \\ R[T_1, \ldots, T_{n+r}] \arrow[u, "h"] \arrow[rr, two heads] & & R[T_{r+1}, \ldots, T_{n+r}] \arrow[u] \end{tikzcd}$$

is cartesian and h is etale. In particular, $J_{f_i} = (h(T_1), \ldots, h(T_r))$ is generated by a regular sequence.

PROOF. Let $x \in \mathrm{Spec}(B)$, $k = k(x)$. By Theorem A.19, the sequence

$$0 \to J/J^2 \otimes k \to \Omega^1_{A/R} \otimes k \to \Omega^1_{B/R} \otimes k \to 0$$

is exact. Let $s_1, \ldots, s_r \in J$ such that they are a basis of $J/J^2 \otimes k$. Then by Nakayama's Lemma they are a minimal generating set of J_x. Since J is of finite type, we can assume, after shrinking if necessary, that they generate J. Let $s_{r+1}, \ldots, s_{n+r} \in A$ such that $ds_1(x), \ldots, ds_{n+r}(x)$ are a basis of $\Omega^1_{A/R} \otimes k$. Shrinking if necessary and using Nakayama's Lemma again, we can assume that sections $ds_1, \ldots, ds_{n+r}$ are a basis of $\Omega^1_{A/R}$. Then the sections $s_1, \ldots, s_{r+n}$ define an etale map $R[T_1, \ldots, T_{r+n}] \to A$ (by Theorem A.24). The sections $s_{r+1}, \ldots, s_{n+r}$ define an etale map $R[T_{r+1}, \ldots, T_{r+n}] \to B$ and since J is generated by $s_1, \ldots, s_r$, we are done. The statement on regularity follows since h is flat (Theorem A.17). □

COROLLARY A.26. *Let $R \to A \to B$ be ring homomorphisms. Suppose that $R \to A$ and $R \to B$ are smooth and that $B = A/J$. Suppose that J is generated by a regular sequence $(x_1, \ldots, x_d)$ (this is always the case locally). Then there exists an isomorphism $A/J^N \simeq B[T_1, \ldots, T_d]/J_0^N$, where $J_0 = (T_1, \ldots, T_d)$. The isomorphism commutes with the map to B sending T_i to zero.*

PROOF. By smoothness of $R \to B$, we can inductively construct maps $f_N : B \to A/J^N$ by filling out the diagram

$$\begin{tikzcd} B \arrow[r, "f_{N-1}"] \arrow[dr, dashed, "f_N"] & A/J^{N-1} \\ R \arrow[u] \arrow[r] & A/J^N \arrow[u, two heads] \end{tikzcd}$$

We can extend f_N to $g_N : B_1 = B[T_1, \ldots, T_d] \to A/J^N$ by sending T_i to x_i. Clearly this map factors through J_0^N. To check that it is an isomorphism we look at the diagram

$$\begin{tikzcd} 0 \arrow[r] & J^N/J^{N-1} \arrow[r] & A/J^N \arrow[r] & A/J^{N-1} \arrow[r] & 0 \\ 0 \arrow[r] & J_0^N/J_0^{N-1} \arrow[u, "\simeq"] \arrow[r] & B_1/J_0^N \arrow[u] \arrow[r] & B_1/J_0^{N-1} \arrow[u, "\simeq"] \arrow[r] & 0 \end{tikzcd}$$

(the map on the left is an isomorphism because the sequence is regular). □

APPENDIX B

Naïve formal schemes

A *formal scheme P over R* is a sequence of closed embeddings $P_0 \subset \cdots \subset P_N \subset P_{N+1} \subset \cdots$ of R-schemes such that $I^{N+1}\mathscr{O}_{P_N} = 0$ and such that $P_N = P_{N+1} \otimes_{R_{N+1}} R_N$. Formal schemes form a category in an obvious way: a morphism between formal schemes $P \to Q$ is a family of compatible maps $P_N \to Q_N$.

Given a R-scheme X, one defines the *formal completion* of X to be the formal scheme with components $X_N = X \otimes R_N$. Clearly this correspondence is functorial. A map $f : P \to Q$ of formal schemes is *smooth* if every component $f_N : P_N \to Q_N$ is smooth. It is a *closed embedding* if every component is one, etc.

Definition B.1. A formal R-scheme X is *smoothable over R* is there exists a smooth formal scheme P together with a closed immersion $X \subset P$ of formal R-schemes. Sometimes we will say that P is a *smoothing of X over R.*

If X is a formal smoothable scheme over R, define

$$\widehat{\Omega}^{\bullet}_{D_Y(X)/R} = \varprojlim_N \Omega^{\bullet}_{D_{Y_N}(X_N)/R_N}.$$

This is a complex of sheaves of R-modules on X. To check that this is independent of Y, we need to study the behaviour of this construction with respect to quasi-isomorphisms. Recall that an inverse system of modules $M_\bullet$ is said to be *Mittag-Leffler* if .... We now recall the Mittag-Leffler criterion for sheaves.

Lemma B.2. *Let X be a topological space, $\mathscr{F}_\bullet$ an inverse system of abelian sheaves on X, and suppose that there exists a basis of open sets of X satisfying*

i) $H^q(V, \mathscr{F}_n) = 0$ *if* $q > 0$,

ii) *The inverse system $H^0(V, \mathscr{F}_n)$ satisfies the Mittag-Leffler condition.*

Then $\mathscr{F}_\bullet$ is $\varprojlim$-acyclic.

Proof. TBA. □

Note that the hypothesis of Lemma B.2 are satisfied by $\mathrm{Fil}^n\,\Omega^i_{D_{Y_\bullet}(X)/R_\bullet}$ for any i and n. Indeed, if $V \subset X$ is an affine open, then i) holds by quasi-coherence. And ii) holds because the reduction maps $\mathrm{Fil}^n\,\Omega^i_{D_{Y_{N+1}}(X)/R_{N+1}} \to \mathrm{Fil}^n\,\Omega^i_{D_{Y_N}(X)/R_N}$ are surjective.

Theorem B.3. *Let X be a formal R-scheme, Y and P two smoothings of X over R. Suppose that $Y \to P$ is a smooth morphism or a closed embedding. Then the natural map*

$$\widehat{\Omega}^{\bullet}_{D_P(X)} \longrightarrow \widehat{\Omega}^{\bullet}_{D_Y(X)}$$

is a filtered R-linear quasi-isomorphism.

Proof. Lemma B.2 implies[1] that the natural maps

$$\varprojlim_N \operatorname{Fil}^n \Omega^\bullet_{D_{Y_N}(X_N)} \to R \varprojlim \operatorname{Fil}^n \Omega^\bullet_{D_{Y_N}(X_N)} \tag{5}$$

are isomorphisms in $D^+(X, R)$. In particular, given another smooth scheme Z over R which admits X as a closed subscheme induces canonical quasi-isomorphisms

$$\operatorname{Fil}^n \widehat{\Omega}^\bullet_{D_Y(X)/R} \to \operatorname{Fil}^n \widehat{\Omega}^\bullet_{D_Z(X)/R}.$$

□

[1]Use the hypercohomology spectral sequence

$$R^i \varprojlim_N \Omega^j_{D_{Y_N}(X)} \implies R^{i+j} \varprojlim_N \Omega^\bullet_{D_{Y_N}(X)}.$$

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