THE UNIVERSITY OF CHICAGO

# TOWARDS THE COHOMOLOGICAL CONSTRUCTION OF BREUIL-KISIN MODULES

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Para mis papás,

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y

mis hermanas,

Eva, Ana y Laura

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> Niña morena y ágil, el sol que hace las frutas,
> el que cuaja los trigos, el que tuerce las algas,
> hizo tu cuerpo alegre, tus luminosos ojos
> y tu boca que tiene la sonrisa del agua.
>
> Eres la delirante juventud de la abeja,
> la embriaguez de la ola, la fuerza de la espiga.
> Mariposa morena dulce y definitiva,
> como el trigal y el sol, la amapola y el agua.

# INTRODUCTION

The purpose of this thesis is to give a construction of certain modules appearing in Kisin's approach [13] to integral $p$-adic Hodge theory and to clarify their relationship to previous theories due to Breuil [3] and Faltings [8].

Let us briefly recall the relevant notions. Let $K/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_K$ and residue field $k$. Let $W = W(k)$ be the ring of Witt vectors and $K_0 = W[1/p]$ its fraction field. Let $e$ be the degree of $K$ over $K_0$. Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of $K$. A *$p$-adic Galois representation* is a pair $(V, \rho)$ consisting of a finite-dimensional $\mathbf{Q}_p$-vector space $V$ and a continuous map $\rho : G_K \to GL(V)$.

Generalizing classical Dieudonné theory [9], Fontaine had the idea of classifying and constructing $p$-adic Galois representations by utilizing semi-linear algebra. The basic structure is the following: a *filtered $\varphi$-module* is a triple $(D, \varphi, \mathrm{Fil})$ consisting of a finite-dimensional $K_0$-vector space $D$, a $K_0$-semi-linear bijection $\varphi : D \to D$ and a filtration $\mathrm{Fil}^\bullet D_K$ on the vector space $D_K = K \otimes D$. Furthermore, Fontaine found a way to relate filtered $\varphi$-modules and $p$-adic Galois representations, as follows. He discovered a topological $K_0$-algebra $B_{\mathrm{cris}}$ equipped with a $K_0$-semi-linear map $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$, a continuous $G_K$-action, and a filtration on $B_{\mathrm{cris}} \otimes_{K_0} K$. Given this ring, one can write a functor from $p$-adic Galois representations to filtered $\varphi$-modules by

$$D_{\mathrm{cris}}(V, \rho) = (B_{\mathrm{cris}} \otimes V)^{G_K}.$$

We say that $(V, \rho)$ is *crystalline* if $\dim_{K_0} D(V, \rho) = \dim_{\mathbf{Q}_p} V$. The functor $D$ restricts to a fully faithful embedding on the full subcategory of crystalline $p$-adic Galois representations [18]. The filtered $\varphi$-modules in the essential image of $D_{\mathrm{cris}}$ is said to be *admissible.*

This reduces the (hard) study of (crystalline) $p$-adic Galois representations to the (easier) study of their associated admissible modules. In fact, it is possible to characterize those fitlered $\varphi$-modules in the essential image of $D_{\mathrm{cris}}$, as follows. Suppose that $D$ is a filtered $\varphi$-module. Define its *Hodge number* $t_H(D) = \sum_{i \in \mathbf{Z}} i \cdot \dim_K \mathrm{gr}^i D_K$. Since $\varphi$ is only semi-linear, its determinant is not well-defined, but it is easy to see that its $p$-adic valuation is. Thus we can define its *Newton number* as $t_N(D) = v_p(\det \varphi)$. Then $D$ is called *weakly admissible* if $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for any subobject $D' \subseteq D$. The so-called Theorem B of $p$-adic Hodge theory is the statement that *admissible = weakly admissible.*

Unfortunately, in applications the above theory is not enough because one needs to understand not only the $p$-adic representation $V$, but also its $G_K$-stable $\mathbf{Z}_p$-lattices. It is natural to wish for an *integral* version of the semi-linear side of the above equivalence. Vaguely speaking, this integral theory would match $G_K$-stable $\mathbf{Z}_p$-lattices of $V$ with certain "lattices" of $D$. There were several attempts to realize this dream, which achieved success under some restrictive hypothesis. In particular, we should mention Fontaine-Laffaille theory [11] and Breuil's theory of $S$-modules [4] (we will recall the definition of $S$ below). However, all these theories required that the ramification of $K$ be "small": $e < p$ (or even $e < p - 1$). This restriction is often not satisfied in applications.

However, in [13] Kisin managed to produce a theory without any such restriction on $e$, albeit with one important caveat. Instead of classifying $G_K$-stable $\mathbf{Z}_p$-lattices of $V$, it classifies $G_{K_\infty}$-stable $\mathbf{Z}_p$-lattices of $V$, where $G_{K_\infty} \subset G_K$ is the absolute Galois group of $K_\infty = \bigcup_n K(\pi^{p^{-n}})$. Nonetheless, this result led to important applications in the study of finite flat group schemes, deformation rings of Galois representations and Shimura varieties among other subjects (see especially [14], [15] and [16]).

Let us now briefly describe the main objects appearing in this theory (which is closely related to Fontaine's theory of $(\varphi, \Gamma)$-modules [10]).

Let $\mathfrak{S}$ be the ring of formal power series in one variable $u$ over $W$ and let $\mathcal{O}$ be the ring of rigid analytic functions on the disk over $K_0$, $\mathcal{O} = \{f \in K_0[[u]] : f \text{ converges for } |u| < 1\}$. We equip both of these rings with the $W$-semi-linear map $\varphi$ that sends $u$ to $u^p$. A filtered $\varphi$-module $(D, \varphi, \mathrm{Fil}^\bullet)$ is *effective* if $\mathrm{Fil}^0 D = D$ (one can always reduce to this case by using Tate twists). A $(\varphi, \mathfrak{S})$-module (resp. $(\varphi, \mathcal{O})$-module) $M$ is said to be of *finite $E$-height* if the cokernel of $\varphi^* M \to M$ is killed by $E(u)^r$ for some $r > 0$. In [13] one finds the construction of a diagram of fully faithful $\otimes$-functors

$$\begin{array}{ccc}
\text{(effective) filtered } \varphi\text{-modules} & \xrightarrow{\ \mathcal{M}\ } & \begin{array}{c}(\varphi, \mathcal{O})\text{-modules of} \\ \text{finite } E\text{-height}\end{array} \\
\big\uparrow & & \big\uparrow {\scriptstyle -\otimes_{\mathfrak{S}} \mathcal{O}} \\
\begin{array}{c}\text{(effective) weakly admissible} \\ \text{filtered } \varphi\text{-modules}\end{array} & \longrightarrow & \begin{array}{c}(\varphi, \mathfrak{S}[1/p])\text{-modules of} \\ \text{finite } E\text{-height}\end{array}
\end{array}$$

Let us next describe the problem addressed in this thesis. Suppose that $X$ is a proper and smooth scheme over $\mathcal{O}_K$ and let $V = H^m_{et}(X_K \otimes \overline{K}, \mathbf{Q}_p)$ be one of its $p$-adic étale cohomology groups. Associated to such a $p$-adic Galois representation there is an (effective) filtered $\varphi$-module where the underlying $K_0$-vector space is given by the corresponding crystalline cohomology group $D = H^m_{\mathrm{cris}}(X_k/W)[1/p]$ (where we write $X_k$ for the base change of $X$ to the residue field $k$). The filtration on $D_K$ is given by the Hodge filtration on $H^m_{dR}(X_K/K)$ via the Berthelot-Ogus isomorphism $H^m_{\mathrm{cris}}(X_k/W) \otimes K \simeq H^m_{dR}(X_K/K)$.

The work in this article arose from thinking about the following question:

**Question.** *How can one construct the $\mathfrak{S}[1/p]$-module that corresponds to $H^m_{\mathrm{cris}}(X_0/W)[1/p]$ directly from $X$?*

This is a natural question to ask. At the same time, having such constructions is often very useful in applications. Questions of this kind (usually subsumed under the heading of *p-adic comparison isomorphisms*) have been studied at length for both the Fontaine-Laffaille theory and Breuil's theory of $S$-modules (see [12], [8] and [4]). In both cases, the corresponding semi-linear object is constructed using crystalline cohomology.

Similarly, our construction also uses crystalline theory. Before explaining the rough outlines of this construction, let us state the main result. Let $\mathcal{M}(D)$ be the $(\varphi, \mathcal{O})$-module corresponding to a filtered $\varphi$-module $D$ as pictured in the above diagram. We prove the following result (the actual result, Proposition 5.5.1, also allows for non-constant coefficients).

**Proposition.** *Let $X/\mathcal{O}_K$ be proper and smooth. There is a bounded complex $\mathcal{M}(X)$ of finite free $(\varphi, \mathcal{O})$-modules on $X_k$ and functorial isomorphisms*

$$\mathcal{M}(H^q_{\mathrm{cris}}(X_k/W)[1/p]) \simeq H^q(X_k, \mathcal{M}(X))$$

*of $(\varphi, \mathcal{O})$-modules.*

Let us now sketch the approach used in proving the theorem. Fix an uniformizer $\pi$ of $K$ over $K_0$ and let $E(u)$ be the corresponding Eisenstein polynomial. Breuil [4] and Faltings [8] both studied crystalline cohomology over a certain nilpotent thickening $S$ of $\mathcal{O}_K$ defined as follows:

$$S := \varprojlim_n W[[u]][E(u)^i/i!]_{i\geq 0}/p^n \subset K_0[[u]].$$

On the other hand, the construction [13] that we want to recreate cohomologically works by looking at the tower of field extensions $K \subset K(\pi^{p^{-1}}) \subset K(\pi^{p^{-2}}) \subset \cdots$. Hence it is natural to look at the sequence of rings $\cdots \subset S_2 \subset S_1 \subset S_0$ where $S_n$ is defined as above with $\varphi^n(E(u))$ in place of $E(u)$. This sequence of rings has the property that:

$$\mathfrak{S} = \bigcap S_n \qquad \text{and} \qquad \mathcal{O} = \bigcap S_n[1/p].$$

The rough idea is then to consider crystalline cohomology of suitable base changes of $X$ over the different $S_n$'s and "paste" these cohomology groups together (after performing some modifications guided by the formulas in [13]) and finally passing to the limit over $n$.

In fact, it is possible to set up this construction on the level of (complexes of) sheaves on $X$ (or rather, its special fiber). If we do this with complexes on $X$ representing crystalline cohomology we obtain the previous construction after taking global sections.

This program is carried out (with one caveat, see below) in this thesis in the case of the $\mathcal{O}$-module $\mathcal{M}(D)$, and it is very likely that the same construction also works in the integral case of $\mathfrak{S}$. However, we have not yet obtained the result in the case of $\mathfrak{S}$, so it is not included in this thesis.

The caveat alluded to above refers to the fact that the ring $S$ as written does not have the right properties needed to make the proofs work. Hence we were led to introduce a slightly larger PD-nilpotent thickening of $\mathcal{O}_K$, which we call $S^{pd}$, which in some aspects is better-behaved than $S$. This new ring does not seem to have been considered previously and we hope that it will be useful in the future in other contexts.

## Contents

In Chapter 1 we define and review the filtered derived category. The only novelty here is that we work with $\varphi$-equivariant objects throughout.

In Chapter 2 we review crystalline cohomology in the form that we need. We check that crystalline cohomology exists as an object in the filtered $\varphi$-equivariant derived category of Chapter 1.

In Chapter 3 we define and prove some basic properties of the new PD-ring $S^{pd}$ mentioned above.

In Chapter 4 we present a new construction of the $\mathcal{O}$-module $\mathcal{M}(D)$ attached to a filtered $\varphi$-module $D$ using the ring $S^{pd}$ introduced in Chapter 3.

In Chapter 5 we lift the constructions of Chapter 4 to the level of sheaves. We tie everything together by defining a functor from certain crystalline coefficients to complexes of $(\varphi, \mathcal{O})$-modules and we check that upon taking cohomology we recover the construction of Chapter 4 and therefore the $\mathcal{O}$-module defined by Kisin [13].

## Notation

In all that follows, $p$ will be an odd prime number, $k$ will be a perfect field of characteristic $p$, $W = W(k)$ will be the ring of Witt vectors, $K_0 = W[1/p]$ the field of fractions, $E(u) \in W[u]$ will be an Eisenstein polynomial, $\mathcal{O}_K = W[u]/E(u)$ the ring of integers of $K = \mathcal{O}_K[1/p]$, a totally ramified extension of $K_0$ of degree $e = \deg E(u)$. Finally we will denote by $\pi$ the image of $u$ in $\mathcal{O}_K$, an uniformizer. We will also write $v$ for the $p$-adic valuation normalized so that $v(p) = 1$. Filtrations on an object $X$ will satisfy $\operatorname{Fil}^0 X = X$ unless something is said to the contrary.

# CHAPTER 1

# FILTERED $(\varphi, S)$-COMPLEXES

The objective of this chapter is to present the filtered derived category of Frobenius-equivariant sheaves on a topological space. The development is parallel to that of the non-equivariant case (e.g. [19, Chapter 1]), except that since we do not know if the category of filtered Frobenius-equivariant sheaves has enough injective objects, we have to work systematically with flasque resolutions.

In Sections 1, 2, and 3 we define the relevant derived categories and check that they can be computed using flasque resolutions. Section 4 contains the definition of the derived pushforward of a map of ringed spaces. Sections 5 and 6 extend these constructions to the case of inverse systems.

## 1.1

Let $X$ be a topological space. Let $S$ be a filtered sheaf of rings on $X$. Then $S$ has a filtration by ideals $\operatorname{Fil}^k S \subseteq S$ ($k \in \mathbf{Z}$) such that $\operatorname{Fil}^k S \cdot \operatorname{Fil}^l S \subseteq \operatorname{Fil}^{k+l} S$. A *filtered $S$-module $E$ on $X$* is a $S$-module $E$ equipped with a decreasing filtration $\operatorname{Fil}^k E \subseteq E$, $k \in \mathbf{Z}$ such that $\operatorname{Fil}^k S \cdot \operatorname{Fil}^l E \subseteq \operatorname{Fil}^{k+l} E$. We will always assume that all filtrations are effective (i.e, $\operatorname{Fil}^0 S = S$, $\operatorname{Fil}^0 E = E$, etc.). Note that in general our filtrations will not be finite.

Let $\varphi : S \to S$ be a morphism of sheaves of rings. A *filtered $(\varphi, S)$-module* is a pair $(E, \varphi)$ consisting of a filtered $S$-module $E$ together with a $\varphi$-semi-linear map $\varphi_E : E \to E$. We will refer to this map as the *Frobenius of $E$* (and we will usually write just $\varphi$ instead of $\varphi_E$).

Let $MF(X, S, \varphi)$ be the category of filtered $(\varphi, S)$-modules on $X$ together with homomorphisms of $S$-modules that preserve the filtration and commute with the Frobeniuses. Let $CF(X, S, \varphi)$ be the category of complexes of filtered $(\varphi, S)$-modules on $X$. We will call them *filtered* $(\varphi, S)$*-complexes.* Let $C^+F(X, S, \varphi)$, $C^-F(X, S, \varphi)$ and $C^bF(X, S, \varphi)$ be the full subcategories of those complexes which are, respectively, bounded below, bounded above, and bounded both above and below.

A *filtered* $\varphi$*-homotopy* between filtered $(\varphi, S)$-complexes is a homotopy of the underlying complexes which is filtration- and $\varphi$-preserving. Let $KF(X, S, \varphi)$ be the category that has the same objects as $CF(X, S, \varphi)$ but the morphisms are filtered $\varphi$-homotopy classes of morphisms in $CF(X, S, \varphi)$. Similarly, we have quotients $K^+F(X, S, \varphi)$, $K^-(F, S, \varphi)$ and $K^bF(X, S, \varphi)$. They are full subcategories of $KF(X, S, \varphi)$.

A filtered $(\varphi, S)$-complex $E$ is *filtered exact* if $H^q \operatorname{Fil}^k E = 0$ for all $q, k \in \mathbf{Z}$. This is a well defined notion for objects of the category $KF(X, S, \varphi)$. We say that a map $f : E \to E'$ of filtered $(\varphi, S)$-complexes is a *filtered* $\varphi$*-quasi-isomorphism* if $\operatorname{Fil}^k f : \operatorname{Fil}^k E \to \operatorname{Fil}^k E'$ are quasi-isomorphisms for every $k \in \mathbf{Z}$.

**Lemma 1.1.1.** *Let* $* = +, -, b,$ *or* $\emptyset$*. The category* $K^*F(X, S, \varphi)$ *is triangulated and the class of filtered* $\varphi$*-quasi-isomorphisms in* $K^*F(X, S, \varphi)$ *is localizing.*

*Proof.* If we forget about $\varphi$, this result is explained in [19, p. 16]. The same proof works in this case. See also [7, Theorem III.4.4]. □

**Definition 1.1.2.** *The filtered* $\varphi$*-derived category of* $(X, S, \varphi)$ *is the localization of the triangulated category* $KF(X, S, \varphi)$ *by the class of filtered* $\varphi$*-quasi-isomorphisms. We denote it by* $DF(X, S, \varphi)$*. Similarly, we define* $D^*F(X, S, \varphi)$ *for* $* = +, -, b$*.*

## 1.2

For any integer $k \in \mathbf{Z}$ we have the functor $\mathrm{gr}^k : K^*F(X,S) \to K^*(X,S)$ defined by sending $K$ to $\mathrm{Fil}^k K/\mathrm{Fil}^{k+1} K$.

**Lemma 1.2.1.** *If $f : E \to E'$ is a filtered quasi-isomorphism in $KF(X,S)$, then $\mathrm{gr}^k f : \mathrm{gr}^k E \to \mathrm{gr}^k E'$ is a quasi-isomorphism in $K(X,S)$. Thus $\mathrm{gr}^k$ induces a triangulated functor*

$$\mathrm{gr}^k : D^*F(X,S) \to D^*(X,S).$$

*Proof.* The fact that $\mathrm{gr}^k$ preserves quasi-isomorphisms is clear from the 5-lemma. The rest is clear. □

We also have the forget-the-filtration functor $forget : K^*F(X,S) \to K^*(X,S)$. This functor is exact and preserves quasi-isomorphisms so it induces a functor

$$forget : D^*F(X,S) \to D^*(X,S).$$

Similarly, the functor $\mathrm{Fil}^k : K^*F(X,S) \to K^*(X,S)$ defined by sending $K$ to $\mathrm{Fil}^k K$ preserves quasi-isomorphisms so it induces a functor

$$\mathrm{Fil}^k : D^*F(X,S) \to D^*(X,S).$$

## 1.3

Let $K$ be an abelian sheaf on $X$. Let $\mathcal{C}(K)$ be the associated Godement sheaf:

$$\mathcal{C}(K)(U) := \prod_{x \in U} K_x.$$

There is an obvious inclusion $K \subseteq \mathcal{C}(K)$. Note that $\mathcal{C}(K)$ is flasque. The association $K \mapsto \mathcal{C}(K)$ is functorial and exact.

**Definition 1.3.1.** *A filtered $(\varphi, S)$-module $E$ is* filtered flasque *if* $\operatorname{Fil}^k E$ *are flasque for every $k \in \mathbf{Z}$.*

**Lemma 1.3.2.** *Given any filtered $(\varphi, S)$-module $K$ there is a filtered flasque $(\varphi, S)$-module $L$ and a map of $(\varphi, S)$-modules $K \to L$ which is a filtered monomorphism (i.e.,* $\operatorname{Fil}^k L \cap K = \operatorname{Fil}^k K$ *for every $k \in \mathbf{Z}$).*

*Proof.* We define a filtration on $\mathcal{C}(K)$ for a filtered sheaf $K$ by

$$\operatorname{Fil}^k \mathcal{C}(K) := \mathcal{C}(\operatorname{Fil}^k K) \subseteq \mathcal{C}(K)$$

If $x \in X$, there is an obvious map $(\varphi^* K)_x = S \otimes_{\varphi, S} K_x \to K_x$. We define a $\varphi$-action on $\mathcal{C}(K)$ via

$$\varphi^*(\mathcal{C}(K)) = S \otimes_{\varphi, S} \prod_{x \in X} K_x \to \prod_{x \in X} S \otimes_{\varphi, S} K_x \to \prod_{x \in X} K_x = \mathcal{C}(K).$$

Thus $\mathcal{C}(K)$ is naturally a filtered $(\varphi, S)$-module, it is filtered flasque and it is easy to check that the map $K \to \mathcal{C}(K)$ is a filtered monomorphism. □

**Corollary 1.3.3.** *The category $MF(X, S, \varphi)$ contains enough filtered flasque $(\varphi, S)$-modules.*

Let $K^+F(X, S, \varphi)_{flasque}$ be the full subcategory of $K^+F(X, S, \varphi)$ spanned by the filtered flasque $(\varphi, S)$-complexes.

**Corollary 1.3.4.** *The natural functor*

$$(FQis)^{-1} K^+F(X, S, \varphi)_{flasque} \to D^+F(X, S, \varphi)$$

*is an equivalence of triangulated categories.*

*Proof.* This follows from Corollary 1.3.3 and the proof of [7, Proposition III.6.4]. □

## 1.4

A *$\varphi$-ringed space* is a triple $(X, S, \varphi)$ consisting of a topological space $X$, a sheaf of rings $S$ on $X$, and a ring homomorphism $\varphi : S \to S$.

Let $(X, S, \varphi)$ and $(Y, T, \psi)$ be $\varphi$-ringed spaces. A morphism $(X, S, \varphi) \to (Y, T, \psi)$ will be a morphism of ringed spaces $(X, S) \to (Y, T)$ such that

$$\begin{array}{ccc} f^{-1}T & \xrightarrow{\psi} & f^{-1}T \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ S & \xrightarrow{\varphi} & S \end{array}$$

commutes.

**Proposition 1.4.1.** *Let $f : (X, S, \varphi) \to (Y, T, \psi)$ be a homomorphism. Then there is a right derived functor*

$$Rf_* : D^+F(X, S, \varphi) \to D^+F(Y, T, \psi)$$

*of $f_*$ such that*

$$\begin{array}{ccc} D^+F(X, S, \varphi) & \xrightarrow{Rf_*} & D^+F(Y, T, \psi) \\ \downarrow & & \downarrow \\ D^+(X, S) & \xrightarrow{Rf_*} & D^+(Y, T) \end{array}$$

$$\begin{array}{ccc} D^+F(X, S, \varphi) & \xrightarrow{Rf_*} & D^+F(Y, T, \psi) \\ \downarrow & & \downarrow \\ D^+F(X, S) & \xrightarrow{Rf_*} & D^+F(Y, T) \\ {\scriptstyle \mathrm{Fil}^k}\downarrow & & \downarrow{\scriptstyle \mathrm{Fil}^k} \\ D^+(X, S) & \xrightarrow{Rf_*} & D^+(Y, T) \end{array} \qquad \begin{array}{ccc} D^+F(X, S, \varphi) & \xrightarrow{Rf_*} & D^+F(Y, T, \psi) \\ \downarrow & & \downarrow \\ D^+F(X, S) & \xrightarrow{Rf_*} & D^+F(Y, T) \\ {\scriptstyle \mathrm{gr}^k}\downarrow & & \downarrow{\scriptstyle \mathrm{gr}^k} \\ D^+(X, S) & \xrightarrow{Rf_*} & D^+(Y, T) \end{array}$$

*commute up to canonical equivalence (the unlabeled arrows are just the forgetful functors). If $f : (X, S, \varphi) \to (Y, T, \psi)$ and $g : (Y, T, \psi) \to (Z, U, \theta)$ are two such morphisms then there is a natural isomorphism of functors $R(g \circ f)_* \simeq Rg_* \circ Rf_*$.*

*Proof.* We start by defining $Rf_*$ as a functor $Rf_* : K^+F(X, S, \varphi) \to K^+F(Y, T, \varphi)$. Given a complex of filtered flasque $(\varphi, S)$-modules $K$, simply put $Rf_*(K) := f_*(K)$. This is naturally a filtered flasque $(T, \psi)$-complex on $Y$. Flasque sheaves are acyclic for $f_*$, so $Rf_*$ factors through the localization and therefore by Corollary 1.3.4 defines a functor $Rf_* : D^+F(X, S, \varphi) \to D^+F(Y, T, \psi)$. The composition rule and the compatibility with the forgetful functor follows immediately from this definition.

For the compatibility with $\operatorname{Fil}^k$ functors, it is enough to observe that if $J$ is a filtered flasque $(\varphi, S)$-complex, then $\operatorname{Fil}^k J$ is flasque for every $k \in \mathbf{Z}$. Then the compatibility follows from the identifications

$$Rf_*(\operatorname{Fil}^k J) \simeq f_*(\operatorname{Fil}^k J) = \operatorname{Fil}^k f_*(J) \simeq \operatorname{Fil}^k Rf_*(J).$$

The case of $\operatorname{gr}^k$ is similar. □

## 1.5

We carry out the previous construction of $D^+F(X, S, \varphi)$ in the case of inverse systems. Let $S_\bullet$ be an inverse system of sheaves of rings on $X$:

$$\cdots \to S_n \to \cdots \to S_1 \to S_0.$$

A *$S_\bullet$-module on $X$* is an inverse system of modules

$$\cdots \to E_n \to \cdots \to E_1 \to E_0$$

where $E_n$ is a sheaf of $S_n$-modules on $X$ and the transition map $E_n \to E_{n-1}$ is over $S_n \to S_{n-1}$. We can give a similar definition for modules equipped with a filtration $\mathrm{Fil}^q E_n$ and maps that preserve the filtration.

Now suppose that $\varphi_\bullet : S_\bullet \to S_{\bullet+1}$ is a map of inverse systems of rings. A *filtered* $(S_\bullet, \varphi_\bullet)$*-module on* $X$ is a filtered $S_\bullet$-module $E_\bullet$ on $X$ together with a $S_\bullet$-semi-linear map $\varphi_\bullet : E_\bullet \to E_{\bullet+1}$.

Let $KF(X, S_\bullet, \varphi_\bullet)$ be the homotopy category. By inverting filtered $\varphi$-quasi-isomorphisms we define $DF(X, S_\bullet, \varphi_\bullet)$.

**Definition 1.5.1.** *A filtered* $(S_\bullet, \varphi_\bullet)$*-module* $E_\bullet$ *on* $X$ *is called* filtered flasque *if* $E_n$ *and* $\mathrm{Fil}^q E_n$ *is flasque on* $X$ *for every* $n, q \in \mathbf{Z}$ *and if furthermore the inverse systems* $H^0(V, \mathrm{Fil}^q E_\bullet)$ *is surjective (i.e. the transition maps are surjective) for every open* $V \subseteq X$*.*

**Lemma 1.5.2.** *Let* $E_\bullet$ *be a filtered* $(\varphi_\bullet, S_\bullet)$*-module on* $X$*. Then there is a filtered flasque* $(\varphi_\bullet, S_\bullet)$*-module* $J_\bullet$ *and a filtered monomorphism* $E_\bullet \to J_\bullet$*.*

*Proof.* Let $E_\bullet$ be a filtered $(\varphi_\bullet, S_\bullet)$-module on $X$. Put $J_n = \bigoplus_{0 \le k \le n} \mathcal{C}(E_k)$. The transition maps of $E_\bullet$ induce map $J_{n+1} \to J_n$ and it is easy to check that the filtration and the $\varphi$ on $E_\bullet$ induce a filtration and a $\varphi$ on $J_\bullet$ and that the map $E_\bullet \to J_\bullet$ induced by the inclusions $E_n \to \mathcal{C}(E_n)$ satisfies the conclusion of the Lemma. □

Let $KF(X, S_\bullet, \varphi_\bullet)_{flasque}$ be the full subcategory of $KF(X, S_\bullet, \varphi_\bullet)$ spanned by the filtered flasque $(\varphi_\bullet, S_\bullet)$-complexes.

**Corollary 1.5.3.** *The natural functor*

$$(FQis)^{-1} K^+ F(X, S_\bullet, \varphi_\bullet)_{flasque} \to D^+ F(X, S_\bullet, \varphi_\bullet)$$

*is an equivalence of categories.*

*Proof.* Again the usual proof [7, Proposition III.6.4] works in this case. □

Let $(X, S_\bullet, \varphi_\bullet)$ and $(Y, T_\bullet, \psi_\bullet)$ be $\varphi_\bullet$-ringed spaces. A morphism $(X, S_\bullet, \varphi_\bullet) \to (Y, T_\bullet, \psi_\bullet)$ will be a morphism of ringed spaces $(X, S_\bullet) \to (Y, T_\bullet)$ such that

$$\begin{array}{ccc} f^{-1}T_\bullet & \xrightarrow{\psi_\bullet} & f^{-1}T_\bullet \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ S_\bullet & \xrightarrow{\varphi_\bullet} & S_\bullet \end{array}$$

commutes.

We have the following counterpart to Proposition 1.4.1.

**Proposition 1.5.4.** *Let $f : (X, S_\bullet, \varphi_\bullet) \to (Y, T_\bullet, \psi_\bullet)$ be a homomorphism. Then there is a right derived functor*

$$Rf_* : D^+F(X, S_\bullet, \varphi_\bullet) \to D^+F(Y, T_\bullet, \psi_\bullet)$$

*of $f_*$ such that*

$$\begin{array}{ccc} D^+F(X, S_\bullet, \varphi_\bullet) & \xrightarrow{Rf_*} & D^+F(Y, T_\bullet, \psi_\bullet) \\ \downarrow & & \downarrow \\ D^+F(X, S_n, \varphi_n) & \xrightarrow{Rf_*} & D^+F(Y, T_n, \psi_n) \end{array}$$

*commutes up to canonical equivalence (the unlabeled arrows are just the forgetful functors). If $f : (X, S_\bullet, \varphi_\bullet) \to (Y, T_\bullet, \psi_\bullet)$ and $g : (Y, T_\bullet, \psi_\bullet) \to (Z, U_\bullet, \theta_\bullet)$ are two such morphisms then there is a natural isomorphism of functors $R(g \circ f)_* \simeq Rg_* \circ Rf_*$.*

*Proof.* The proof is similar to that of Proposition 1.4.1, using Corollary 1.5.3 instead of Corollary 1.3.4. □

## 1.6

**Lemma 1.6.1.** *A filtered flasque $(S_\bullet, \varphi_\bullet)$-module on $X$ is acyclic for $\varprojlim$.*

*Proof.* This follows from [6, Proposition 13.3.1]. □

Let $E_\bullet$ be a filtered $(\varphi_\bullet, S_\bullet)$-module. Let $S_\infty = \varprojlim S_n$. Let $\varphi_\infty : S_\infty \to S_\infty$ be the ring homomorphism induced by $\varphi_\bullet$. Then $\varprojlim E_n$ is naturally a $(\varphi_\infty, S_\infty)$-module via

$$S_\infty \otimes_{\varphi_\infty} \varprojlim E_n \to \varprojlim S_n \otimes_{\varphi_n} E_n \to \varprojlim E_{n+1}.$$

Hence we have a functor $\varprojlim : K(X, S_\bullet, \varphi_\bullet) \to K(X, S_\infty, \varphi_\infty)$. This functor is left exact and filtered flasque systems are acyclic for it by Lemma 1.6.1. Thus its derived functors exist:

$$R \varprojlim : DF(X, S_\bullet, \varphi_\bullet) \to DF(X, S_\infty, \varphi_\infty).$$

*Remark* 1.6.2. Note that actually we can define $R\varprojlim$ on the whole derived category because flasque systems of complexes are acyclic for $\varprojlim$ even if they are not necessarily bounded.

Since we can find a bounded below resolution for a bounded below system (for example using the above Godement resolutions), the above functor restricts to a functor

$$R \varprojlim : D^+F(X, S_\bullet, \varphi_\bullet) \to D^+F(X, S_\infty, \varphi_\infty).$$

**Lemma 1.6.3.** *Let $f : (X, S, \varphi) \to (Y, T, \psi)$ be a map of $\varphi_\bullet$-ringed topoi.*

$$\begin{array}{ccc} D^+F(X, S_\bullet, \varphi_\bullet) & \xrightarrow{R\varprojlim} & D^+F(X, S_\infty, \varphi_\infty) \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\ D^+F(Y, T_\bullet, \psi_\bullet) & \xrightarrow{R\varprojlim} & D^+F(Y, T_\infty, \psi_\infty) \end{array}$$

*Proof.* Let $E_\bullet$ be a filtered $(\varphi_\bullet, S_\bullet)$-complex. Then we can construct a filtered flasque resolution $E_\bullet \to J_\bullet$ as in Lemma 1.5.2. Note that

$$\varprojlim J_\bullet = \prod_{k \geq 0} \mathcal{C}(E_k)$$

is again a flasque $(\varphi_\infty, S_\infty)$-complex. In particular we obtain the following chain of natural identifications:

$$Rf_*(R\varprojlim E_\bullet) \simeq Rf_*(\varprojlim J_\bullet) \simeq f_*(\varprojlim J_\bullet) \simeq \varprojlim f_*(J_\bullet) \simeq R\varprojlim f_*(J_\bullet) \simeq R\varprojlim Rf_*(E_\bullet)$$

(where the next-to-last isomorphism comes from the fact that $f_*(J_\bullet)$ is filtered flasque). $\square$

# CHAPTER 2

# CRYSTALLINE COHOMOLOGY

In this chapter we review some results in crystalline cohomology. We need to know that crystalline cohomology of a filtered Frobenius crystal (see Definition 2.1.1) exists as an object of the filtered $\varphi$-equivariant derived category defined in Chapter 1 (Proposition 2.2.2). This result is slightly awkward to phrase because the Frobenius pullback is only defined for a crystal, and the category of crystals is not abelian so it is not so simple to consider its derived category. Instead, given a fixed crystal on our scheme, we verify that we can compute its crystalline cohomology using explicit Čech-de Rham complexes that have the structure we need.

## 2.1

Let $S$ be a filtered ring. Assume that $\operatorname{Fil}^1 S$ is a PD-ideal, that all the $\operatorname{Fil}^n S$ are sub-PD-ideals, that $S$ is complete with respect to the topology defined by the ideals $\operatorname{Fil}^n S$ and the $p$-adic topology, and also that each $\operatorname{Fil}^n S$ is $p$-adically complete. Moreover, assume that for very $n \geq 0$ there is some $N \gg 0$ such that $(\operatorname{Fil}^1 S)^{[N]} \subseteq \operatorname{Fil}^n S$. In particular $\operatorname{Fil}^1 S / \operatorname{Fil}^n S$ is a nilideal (every element is nilpotent). We also assume that $S$ is equipped with a lift $\varphi$ of the Frobenius modulo $p$.

Let $X$ be a scheme over $S$. Suppose that the PD-structure of $\operatorname{Fil}^1 S$ extends to $X$. The *PD-nilpotent crystalline site of* $X/S$ is defined as follows. The objects are 4-tuples $(U, T, i, \delta)$ where $U$ is an open subscheme of $X$, $T$ is a $S$-scheme, $i : U \to T$ is a closed embedding and $\delta$ is a PD-nilpotent structure on $\mathcal{I} = \ker(\mathcal{O}_T \to \mathcal{O}_U)$ (recall that this means that $\mathcal{I}^{[n]} = 0$ for some $n \gg 0$). A map $(U, T, i, \delta) \to (U', T', i', \delta')$ is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{i} & T \\ \downarrow & & \downarrow \\ U' & \xrightarrow{i'} & T' \end{array}$$

where $U \to U'$ is an open embedding and $T \to T'$ is a PD-morphism over $S$. It is easy to see that this defines a Grothendieck topology. We denote the corresponding topos by $(X/S)_{\text{cris}}$. Given a sheaf $\mathcal{E}$ on it, we can consider its cohomology groups, which we denote by

$$H^*_{\text{cris}}(X/S, \mathcal{E}).$$

For the definition of crystals and other properties of the crystalline site we refer to [1].

**Definition 2.1.1.** *Let $X$ be a $S$-scheme to which the PD-structure extends. Let $X_0 = X \otimes \mathbf{Z}/p$. A* filtered $\varphi$-crystal *on $X/S$ is a pair $(\mathcal{E}, \varphi)$ consisting of a filtered crystal $\mathcal{E}$ on $X/S$ and a module homomorphism $\varphi : \varphi^*\mathcal{E}_0 \to \mathcal{E}_0$, where $\mathcal{E}_0$ is the crystal obtained by restriction on $X_0/S$ and $\varphi^*\mathcal{E}$ is the pullback under the Frobenius of $X_0/S$. We require that $\varphi$ induces an isomorphism $\varphi[1/p] : \varphi^*\mathcal{E}_0[1/p] \xrightarrow{\simeq} \mathcal{E}_0[1/p]$ in the isogeny category.*

A morphism of filtered $\varphi$-crystals $(\mathcal{E}, \varphi) \to (\mathcal{E}', \varphi')$ is a morphism of the underlying crystals that respects the filtration on $\mathcal{E}$ and such that its reduction mod $p$ commutes with $\varphi$ and $\varphi'$.

## 2.2

In this section we will assume that the ring $S$ is equipped with a lift $\varphi : S \to S$ of Frobenius mod $p$. Furthermore, we assume that $X/S$ is separated[1]. An *embedding system* for $X/S$

1. This is not necessary: if $X/S$ is not separated, one can use affine hypercovers to compute crystalline cohomology.

consists of a 4-tuple $(U_\bullet, Z_\bullet, i_\bullet, \varphi_\bullet)$ where $U_\bullet = \{U_n\}$ is a finite open affine cover of $X$, each $Z_n$ is an affine smooth $S$-scheme, $i_n : U_n \to Z_n$ is a closed embedding and $\varphi_n : Z_n \to Z_n$ is a map over $\varphi : S \to S$ that lifts Frobenius mod $p$. Clearly such embedding systems exist. We will generally denote the embedding system just by $Z_\bullet$. Let $J_I$ be the ideal of $U_I$ inside $Z_I$ and let $Z_{I,n}$ be the reduction of $Z_I$ modulo $\mathrm{Fil}^n S$.

Let $Z_\bullet$ be an embedding system for $X/S$. For each subset of indices $I = \{i_0 < \cdots < i_n\}$, let

$$U_I = U_{i_0} \cap \cdots \cap U_{i_n}, \qquad Z_I = Z_{i_0} \times_S \cdots \times_S Z_{i_n},$$

Let $i_I : U_I \to Z_I$ be the induced closed embedding. Let $D_I$ be the PD-nilpotent PD-envelope of $U_I$ inside $Z_I$. We define a complex $C(Z_\bullet, \mathcal{E}) = C(U_\bullet, Z_\bullet, \varphi_\bullet, \mathcal{E})$ as the total complex associated to the double complex

$$C^{pq} = \varprojlim_{n,m} \prod_{|I|=p} \mathcal{E}(U_I, D_I/J_I^{[n]}) \otimes \Omega^q_{Z_{I,m}/(S/\mathrm{Fil}^m S)},$$

where the differential in the $p$-direction is the Čech differential and the one in the $q$-direction is the de Rham differential. This is a filtered $(\varphi, S)$-complex on $X$.

A *morphism of embedding systems* $(U_\bullet, Z_\bullet, \varphi_\bullet) \to (U'_\bullet, Z'_\bullet, \varphi'_\bullet)$ is a map between the $Z$'s that restricts to open embeddings between the $U$'s and commutes with $\varphi$ and $\varphi'$.

**Proposition 2.2.1.** *Let $Z_\bullet \to Z'_\bullet$ be a morphism of embedding system. Then the induced map*

$$C(Z'_\bullet, \mathcal{E}) \to C(Z_\bullet, \mathcal{E})$$

*is a filtered $\varphi$-quasi-isomorphism.*

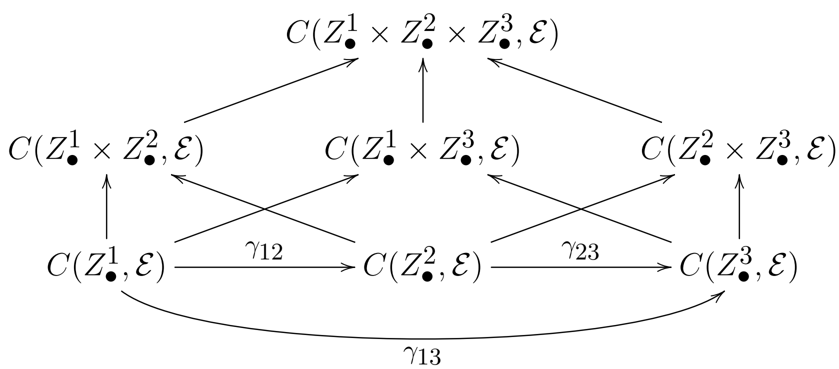
*Proof.* Since the map is $\varphi$-equivariant, it is enough to check that it is a filtered quasi-isomorphism. This is the filtered Poincaré Lemma of [2, Theorem 7.2]. □

**Proposition 2.2.2.** *Suppose that $X$ is a proper $S/\operatorname{Fil}^1 S$-scheme and let Let $(\mathcal{E}, \varphi)$ be a filtered $\varphi$-crystal on $X/S$. Then there is well-defined object in $D^b F(X, S, \varphi)$, unique up to canonical isomorphism. This object can be represented by a bounded complex of finite filtered projective $S$-modules.*

*Proof.* If $Z^1_\bullet$ and $Z^2_\bullet$ are two embedding systems, we can consider their product $Z^1_\bullet \times Z^2_\bullet$ given by the open covering $U^1_i \cap U^2_j$ and the smooth $S$-schemes given by $Z^1_i \times_S Z^2_j$ (with the obvious closed embeddings). Then we obtain a diagram

$$C(Z^1_\bullet, \mathcal{E}) \xrightarrow[\simeq]{qis} C(Z^1_\bullet \times Z^2_\bullet, \mathcal{E}) \xleftarrow[\simeq]{qis} C(Z^2_\bullet, \mathcal{E})$$

of filtered $\varphi$-quasi-isomorphisms. Let $\gamma_{12} : C(Z^1_\bullet, \mathcal{E}) \xrightarrow{\simeq} C(Z^2_\bullet, \mathcal{E})$ be the corresponding isomorphism in the derived category. Next suppose that $Z^1_\bullet$, $Z^2_\bullet$ and $Z^3_\bullet$ are three embedding systems. We can use the projections from the product $Z^1_\bullet \times Z^2_\bullet \times Z^3_\bullet$ to obtain a commutative diagram




It follows that $\gamma_{23} \circ \gamma_{12} = \gamma_{13}$ (cocycle condition) and therefore we see that we have a well-defined object in $D^b F(X, S, \varphi)$.

The second part of the statement was proven by Faltings [8, Theorem 14]. □

**Definition 2.2.3.** *Given a filtered $\varphi$-crystal $(\mathcal{E}, \varphi)$ on $X/S$, we will denote by $Ru_{X/S*}(\mathcal{E}, \varphi)$ the above defined object of $D^+ F(X, S, \varphi)$. We will also denote $R\Gamma(X, Ru_{X/S*}(\mathcal{E}, \varphi))$ by $R\Gamma_{\text{cris}}(X/S, \mathcal{E}, \varphi)$. Both are clearly functorial on the category of filtered $\varphi$-crystals.*

## 2.3

From now on, we will assume that $S$ is equipped with a map $S \to \mathcal{O}_K$ with kernel $\mathrm{Fil}^1 S$. Suppose that $X/\mathcal{O}_K$ is smooth and proper. Let $X_n = X \otimes \mathbf{Z}/p^{n+1}$ and let $\mathcal{E}_n$ be the restriction of $\mathcal{E}$ to $X_n/(S/p^{n+1})$. By the previous part there are well-defined objects $Ru_{X/S*}(\mathcal{E}, \varphi) \in D^+F(X, S, \varphi)$ and $Ru_{X_n/(S/p^{n+1})*}(\mathcal{E}_n, \varphi_n) \in D^+F(X_n, S/p^{n+1}, \varphi_n)$ representing crystalline cohomology.

**Definition 2.3.1.** *Let $X/\mathcal{O}_K$ and $(\mathcal{E}, \varphi)$ be as above. We define*

$$Ru_{\mathcal{X}/S*}(\mathcal{E}, \varphi) := R \varprojlim Ru_{X_n/(S/p^{n+1})*}(\mathcal{E}_n, \varphi_n) \in D^+F(X_0, S, \varphi).$$

**Lemma 2.3.2.** *The restriction map $R\Gamma_{\mathrm{cris}}(X/S, \mathcal{E}, \varphi) \to R\Gamma_{\mathrm{cris}}(\mathcal{X}/S, \mathcal{E}, \varphi)$ is an isomorphism in $D^+F(S, \varphi)$.*

*Proof.* Since the map $R\Gamma_{\mathrm{cris}}(X/S, \mathcal{E}, \varphi) \to R\Gamma_{\mathrm{cris}}(\mathcal{X}/S, \mathcal{E}, \varphi)$ is $\varphi$-equivariant, it is enough to prove that it is a filtered quasi-isomorphism. This is contained in [8, Appendix]. There Faltings constructs a bounded complex of finite filtered projective $S$-modules and a quasi-isomorphism $L \to R\Gamma_{\mathrm{cris}}(X/S, \mathcal{E})$. By the base change theorem in crystalline cohomology it follows that the map $L/p^{n+1} \to R\Gamma_{\mathrm{cris}}(X_n/(S/p^{n+1}), \mathcal{E}_n)$ is an isomorphism. Since $S$ is $p$-adically complete, we see that $L$ is itself $p$-adically complete and thus

$$R\Gamma_{\mathrm{cris}}(X/S, \mathcal{E}) \simeq L \simeq \varprojlim L/p^{n+1} \simeq R \varprojlim R\Gamma_{\mathrm{cris}}(X_n/(S/p^{n+1}), \mathcal{E}_n) \simeq R\Gamma(\mathcal{X}/S, \mathcal{E}).$$

Since the pieces of the filtration on $S$ are all $p$-adically complete, it follows that it is a filtered isomorphism. $\square$

# CHAPTER 3

# THE RING $S^{pd}$

Traditionally, when working with crystalline cohomology over very ramified bases (i.e., when $e \geq p$), the PD-envelope of $W[u] \to \mathcal{O}_K$, $u \mapsto \pi$ (normally denoted simply by $S$) has been used as the base in order to apply crystalline theory. This is the ring that is used by Breuil (for example [3]) and Faltings [8] (where the notation $R_V$ is used for this ring). However, we could not prove our theorems using this ring. For this reason we were led to introduce a substitute of $S$, a new PD-nilpotent thickening of $\mathcal{O}_K$ that does not seem to have been considered previously. This new ring, which we call $S^{pd}$, has better properties than the usual $S$. In particular, Corollary 3.3.3 which will be crucially used in the proof of Proposition 5.4.3 does not hold for the usual $S$.

## 3.1

In this section we define $S^{pd}$ and establish its basic properties. The computations are somewhat long; however, all we are doing is estimating the $p$-divisibility of the coefficients of the change of basis matrix between the bases $\{u^i\}_{i\geq 0}$ and $\{u^i P(u)^j\}_{0\leq i<\deg P, j\geq 0}$ of the ring $W[u]$ (here $P(u)$ is a fixed polynomial).

We will use the following notation: if $x$ is any real number, then

$$\lfloor x \rfloor' := \max\{0, \lfloor x \rfloor\}.$$

Given an integer $i$, we let $q(i)$ be the quotient of $i$ by $e = \deg E(u)$. Define a ring $S^{pd} = S^{pd}(E(u))$ as follows:

$$S^{pd} = S^{pd}(E(u)) = \left\{ \sum_{i=0}^{\infty} a_i u^i : v(a_i) + \left\lfloor \frac{q(i)-1}{p-1} \right\rfloor' \geq 0 \text{ for all } i \geq 0 \right\}.$$

Equivalently, we could use the usual floor function and demand that the first $e$ coefficients $a_0, \ldots, a_{e-1}$ be $p$-adic integers.

**Lemma 3.1.1.** *The set $S^{pd} \subset K_0[[u]]$ is a subring.*

*Proof.* Let $x = \sum a_i u^i$, $y = \sum b_j u^j$ be two elements in $S^{pd}$. That $x + y \in S^{pd}$ follows from $v(a_i + b_i) \geq \min\{v(a_i), v(b_i)\}$. To prove that $xy \in S^{pd}$ it is enough to see that

$$v(a_i b_j) + \left\lfloor \frac{q(i+j)-1}{p-1} \right\rfloor' \geq 0$$

for all $i, j \geq 0$. If $i + j < e$ then $i < e$ and $j < e$ and the claim is trivial. If $i + j \geq e$, then we can remove the prime in the above condition and the inequality follows from $q(i) + q(j) \leq q(i+j)$. □

**Lemma 3.1.2.** *Let $P(u) \in W[u]$ be a monic polynomial of degree $d$. Suppose that every coefficient other than the leading one is divisible by $p$. Let $j$ be a non-negative integer. Write*

$$P(u)^j = \sum_{k=0}^{dj} b_{jk} u^k$$

*for some $b_{jk} \in W$. Then $v(b_{jk}) \geq j - \lfloor k/d \rfloor$ if $j \geq \lfloor k/d \rfloor$.*

*Proof.* Let $i \geq 0$. The coefficient $b_{jk}$ can only fail to have valuation $v(b_{jk}) \geq i$ if it is possible to write $u^k$ as a product of powers of $u$ involving $u^d$ at least $j - i + 1$ times (so that at most $i - 1$ factors with coefficients divisible by $p$ remain in the corresponding product). That is, $v(b_{jk}) \geq i$ if $\lfloor k/d \rfloor < j - i + 1$. Setting $i = j - \lfloor k/d \rfloor$ gives the result. □

**Lemma 3.1.3.** *Let $P(u) \in W[u]$ be a monic polynomial of degree $d$. Suppose that every coefficient other than the leading one is divisible by $p$. Let $0 \leq c < 1$. For each $0 \leq i < d$ and $j \geq 0$, let $a_{ij} \in K_0$ be such that $v(a_{ij}) \geq -cj$ for $j \gg 0$. Then*

$$\sum_{j=0}^{\infty} \sum_{i=0}^{d-1} a_{ij} u^i P(u)^j$$

*converges to an element of $K_0[[u]]$.*

*Proof.* It is enough to prove the lemma for series of the form

$$s = \sum_{j=0}^{\infty} a_j P(u)^j.$$

By Lemma 3.1.2, we can write

$$s = \sum_{j=0}^{\infty} a_j \sum_{k=0}^{dj} b_{jk} u^k = \sum_{k=0}^{\infty} \left( \sum_{j=\lfloor k/d \rfloor}^{\infty} a_j b_{jk} \right) u^k$$

for some $b_{jk} \in W$ such that $v(b_{jk}) \geq j - \lfloor k/d \rfloor$ whenever $j \geq \lfloor k/d \rfloor$. Hence

$$\lim_{j \to \infty} v(a_j b_{jk}) = \lim_{j \to \infty} v(a_j) + v(b_{jk}) \geq \lim_{j \to \infty} -cj + j - \lfloor k/d \rfloor = \infty$$

so the series in the brackets converges. □

**Lemma 3.1.4.** *Let $P(u) \in W[u]$ be a monic polynomial of degree $d$. Suppose that every coefficient other than the leading one is divisible by $p$. Let $k$ be a non-negative integer. Write*

$$u^i = \sum_{j=0}^{\lfloor k/d \rfloor} \sum_{k=0}^{d-1} c_{ijk} u^k P(u)^j.$$

*for some $c_{ijk} \in W$. Then $v(c_{ijk}) \geq \lfloor i/d \rfloor - j$ if $i \geq jd$.*

*Proof.* The proof is by induction on $i$. It holds for $i = jd$ because every $c_{ijk}$ has non-negative valuation. For the inductive step we compute the coefficients for $u^{i+1}$ in terms of the coefficients for $u^i$. Write $P(u) = u^d + b_{d-1}u^{d-1} + \cdots + b_0$, where $b_i \in pW$.

$$\begin{aligned}
uu^i &= u \sum_{j=0}^{\lfloor i/d \rfloor} \sum_{k=0}^{d-1} c_{ijk} u^k P(u)^j = \sum_{j=0}^{\lfloor i/d \rfloor} \sum_{k=0}^{d-1} c_{ijk} u^{k+1} P(u)^j \\
&= \sum_{j=0}^{\lfloor i/d \rfloor} \sum_{k=1}^{d-1} c_{ij(k-1)} u^k P(u)^j + \sum_{j=0}^{\lfloor i/d \rfloor} c_{ij(d-1)} u^d P(u)^j \\
&= \sum_{j=0}^{\lfloor i/d \rfloor} \sum_{k=1}^{d-1} c_{ij(k-1)} u^k P(u)^j + \sum_{j=0}^{\lfloor i/d \rfloor} c_{ij(d-1)} P(u)^{j+1} - \sum_{j=0}^{\lfloor i/d \rfloor} \sum_{k=0}^{d-1} c_{ij(d-1)} b_k u^k P(u)^j.
\end{aligned}$$

Comparing coefficients we find the following table:

$$c_{(i+1)jk} = \begin{cases} c_{i0(d-1)} b_0 & \text{if } j = k = 0, \\ c_{ij(d-1)} b_0 + c_{i(j-1)(d-1)} & \text{if } 0 < j \le \lfloor i/d \rfloor, k = 0, \\ c_{i\lfloor i/d \rfloor (d-1)} & \text{if } j = \lfloor i/d \rfloor + 1, k = 0, \\ c_{ij(k-1)} - c_{ij(d-1)} b_k & \text{if } 0 < k \le d-1. \end{cases}$$

Next we check that if the conclusion of the lemma holds for the $c_{ijk}$ (for any $j$ and $k$ and $i \ge jd$, then it holds for $c_{(i+1)jk}$. We have to check three cases (the third case above doesn't happen occur when $i \ge jd$).

Firstly, suppose that $j = k = 0$. Then

$$v(c_{(i+1)jk}) = v(c_{ijk} b_0) \ge v(c_{ijk}) + 1 \ge \lfloor i/d \rfloor - j + 1 \ge \lfloor (i+1)/d \rfloor - j.$$

Secondly, suppose that $k = 0$ and $0 < j \leq \lfloor i/d \rfloor$. Then

$$\begin{aligned} v(c_{(i+1)jk}) &\geq \min\{v(c_{ij(d-1)}b_0), v(c_{i(j-1)(d-1)})\} \\ &\geq \min\{\lfloor i/d \rfloor - j + 1, \lfloor i/d \rfloor - (j-1)\} \\ &\geq \lfloor (i+1)/d \rfloor - j. \end{aligned}$$

Thirdly, suppose that $k > 0$ and $j$ is arbitrary. Then we will consider two cases. If $\lfloor (i+1)/d \rfloor = \lfloor i/d \rfloor$, then

$$\begin{aligned} v(c_{(i+1)jk}) &\geq \min\{v(c_{ij(k-1)}), v(c_{ij(d-1)}b_0)\} \\ &\geq \min\{\lfloor i/d \rfloor - j, \lfloor i/d \rfloor - j + 1\} \\ &= \lfloor (i+1)/d \rfloor - j. \end{aligned}$$

Finally, suppose that $\lfloor (i+1)/d \rfloor = \lfloor i/d \rfloor + 1$. Then $i + 1$ is a multiple of $d$. As before, we have

$$v(c_{(i+1)jk}) \geq \min\{v(c_{ij(k-1)}), v(c_{ij(d-1)}b_0)\}$$

If $v(c_{ij(k-1)}) \geq v(c_{ij(d-1)}b_k)$ we are done because in this case

$$v(c_{(i+1)jk}) \geq v(c_{ij(d-1)}b_0) \geq \lfloor i/d \rfloor - j + 1 \geq \lfloor (i+1)/d \rfloor - j.$$

Otherwise, we have

$$v(c_{(i+1)jk}) \geq v(c_{ij(k-1)}).$$

Repeating the argument, we see that one of two things can happen. Either there is an

$0 \le r < k < d$ such that

$$v(c_{(i-r)j(k-r)}) \geq v(c_{(i-r)j(d-1)}b_{k-r+1})$$

in which case we are done because

$$v(c_{ijk}) \geq v(c_{(i-r)j(d-1)}b_0) \geq \lfloor (i-r)/d \rfloor - j + 1 = \lfloor i/d \rfloor - j + 1 \geq \lfloor (i+1)/d \rfloor - j$$

(where $\lfloor (i-r)/d \rfloor = \lfloor i/d \rfloor$ because $i \equiv -1 \pmod{d}$ and $0 < r < d$).

The other possibility is that no such $r$ exists in which case we deduce that

$$v(c_{ijk}) \geq v(c_{(i-k+1)j1}),$$

in which case we easily check using the table that the conclusion holds (again use that $\lfloor (i-k)/d \rfloor = \lfloor i/d \rfloor$ because $0 \le k < d$ and $i \equiv -1 \pmod{d}$). □

**Lemma 3.1.5.** *Let $P(u) \in W[u]$ be a monic polynomial of degree $d$. Suppose that every coefficient other than the leading one is divisible by $p$. Suppose that $d < e(p-1)$. Let $a_{jk} \in K_0$ be such that*

$$v(a_{jk}) + \left\lfloor \frac{q(jd) - 1}{p - 1} \right\rfloor' \geq 0. \tag{3.1}$$

*Then the series*

$$\sum_{j=0}^{\infty} \sum_{k=0}^{d-1} a_{jk} u^k P(u)^j$$

*converges and defines an element of $S^{pd}$.*

*Proof.* For convergence, we use Lemma 3.1.3, for which we have to prove the existence of a suitable constant $0 \le c < 1$ such that $v(a_{jk}) \geq -cj$ for $j \gg 0$. From (3.1), it is enough to find such a $c$ such that

$$0 \leq \left\lfloor \frac{q(jd)-1}{p-1} \right\rfloor \leq cj$$

for $j \gg 0$. We have

$$\left\lfloor \frac{q(jd)-1}{p-1} \right\rfloor \leq \frac{q(jd)-1}{p-1} = j\left(\frac{d-e/j}{e(p-1)}\right).$$

Hence we can take $c = d/e(p-1)$. Next, let us verify that the resulting element of $K_0[[u]]$ actually belongs to $S^{pd}$. By Lemma 3.1.2, we can write

$$\begin{aligned} \sum_{j=0}^{\infty}\sum_{k=0}^{d-1} a_{jk}u^k P(u)^j &= \sum_{j=0}^{\infty}\sum_{k=0}^{d-1} a_{jk}u^k \sum_{i=0}^{jd} b_{ji}u^i \\ &= \sum_{k=0}^{d-1} u^k \sum_{i=0}^{\infty}\left(\sum_{j=\lfloor i/d\rfloor}^{\infty} a_{jk}b_{ji}\right)u^i \end{aligned}$$

Suppose first that $i \geq e$. Then $jd \geq e$ as well and the valuation of the coefficient in brackets is:

$$v(a_{jk}b_{ji}) \geq -\left\lfloor \frac{q(jd)-1}{p-1} \right\rfloor + j - \lfloor i/d \rfloor$$

if $j \geq \lfloor i/d \rfloor$. We need to check it is at least $-\lfloor (q(i)-1)/(p-1) \rfloor$. Computing, we see that this follows from

$$-\frac{jd}{e} + (p-1)j - \left\lfloor \frac{i}{d} \right\rfloor (p-1) + \frac{i}{e} \geq \left(-\frac{d}{e} + p - 1\right)\left(j - \left\lfloor \frac{i}{d} \right\rfloor\right)$$

being non-negative. Since $j \geq \lfloor i/d \rfloor$ and $d < e(p-1)$, we are done.

If $0 \leq i < e$, the calculation is even easier. In this case $jd < e$, so that $v(a_{jk}) \geq 0$ and we need to prove that $j - \lfloor i/d \rfloor \geq 0$, which we are assuming. □

**Lemma 3.1.6.** *Let $P(u) \in W[u]$ be a monic polynomial of degree $d$. Suppose that every coefficient other than the leading one is divisible by $p$. Suppose that $d < e(p-1)$. Then* *every elements $s \in S$ can be written in the form* (3.1).

*Proof.* Let $s = \sum_{i=0}^{\infty} a_i u^i$ be an element of $S^{pd}$. Then, by Lemma 3.1.4, we can write

$$s = \sum_{i=0}^{\infty} a_i \sum_{j=0}^{\lfloor i/d \rfloor} \sum_{k=0}^{d-1} c_{ijk} u^k P(u)^j = \sum_{j=0}^{\infty} \sum_{k=0}^{d-1} \left( \sum_{i=dj}^{\infty} a_i c_{ijk} \right) u^k P(u)^j,$$

where $v(c_{ijk}) \geq \lfloor i/d \rfloor - j$ if $i \geq jd$. Suppose first that $jd \geq e$. We need to check that

$$v(a_i c_{ijk}) \geq - \left\lfloor \frac{q(jd) - 1}{p - 1} \right\rfloor$$

when $i \geq jd$. We see that

$$v(a_i c_{ijk}) = v(a_i) + v(c_{ijk}) \geq - \left\lfloor \frac{q(i) - 1}{p - 1} \right\rfloor + \lfloor i/d \rfloor - j.$$

Clearing denominators we are left to see that

$$\frac{jd}{e} - \frac{i}{e} + (p-1) \left\lfloor \frac{i}{d} \right\rfloor - j(p-1) = \left( \frac{d}{e} - (p-1) \right) \left( j - \left\lfloor \frac{i}{d} \right\rfloor \right)$$

is non-negative. Since $d < (p-1)e$ and $i \geq jd$, we are done. If $0 \leq jd < e$, the calculation is trivial and is left to the reader. □

## 3.2

In this section we will equip $S^{pd}$ with a PD-structure on (the completion of) the ideal generated by $E(u)$. In order to do so, first we rewrite the elements of the ring in a way that clearly involves some divided powers.

Choose elements $r_k \in W$ such that

$$v(r_k) = \left\lfloor \frac{k-1}{p-1} \right\rfloor'.$$

By Lemma 3.1.6, we can describe $S$ as the ring

$$S^{pd} = \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{e-1} a_{jk} u^j \frac{E(u)^k}{r_k} : a_{jk} \in W \right\}. \tag{3.2}$$

Define a filtration on $S^{pd}$ by

$$\mathrm{Fil}^i S^{pd} = \left\{ \sum_{k=i}^{\infty} \sum_{j=0}^{e-1} a_{jk} u^j \frac{E(u)^k}{r_k} : a_{jk} \in W \right\}.$$

**Corollary 3.2.1.** *There is a natural map $S^{pd} \to \mathcal{O}_K$, $u \mapsto \pi$, surjective and with kernel is* $\mathrm{Fil}^1 S$*. Moreover, this kernel has divided powers (necessarily unique).*

*Proof.* The first statement is obvious from the description (3.2). For the assertion about divided powers, it is enough to check that the elements $E(u)^k/r_k$, $k > 0$, have divided powers, i.e, that

$$\frac{1}{j!} \left( \frac{E(u)^k}{r_k} \right)^j$$

belongs to $S^{pd}$ for every $j > 0$. Equivalently, that

$$v(j!) + jv(r_k) \leq v(r_{kj}).$$

If $s_j$ is the sum of the $p$-adic digits of $j$, then we have:

$$v(j!) + jv(r_k) = \frac{j - s_j}{p-1} + j \left\lfloor \frac{k-1}{p-1} \right\rfloor \leq \frac{j-1}{p-1} + \frac{jk-j}{p-1} \leq \frac{jk-1}{p-1}.$$

Since the left hand side is an integer, it follows that

$$v(j!) + jv(r_k) \leq \left\lfloor \frac{jk-1}{p-1} \right\rfloor$$

and we are done. □

## 3.3

The main result of this section is Corollary 3.3.3, which holds for $S^{pd}$ but does not hold for the ring $S$ of Breuil [3] (see Remark below). This result is used to prove the important Proposition 5.4.3.

*Remark* 3.3.1. Let $S$ be the PD-completion of the PD-envelope of the map $W[u] \to \mathcal{O}_K$. This is the ring denoted by $R_V$ in [8]. If $\operatorname{Fil}^1 S$ denotes the (closure) of the ideal generated by the divided powers $E(u)^{[n]}$, $n > 0$, then it is immediate that $S/\operatorname{Fil}^1 S \simeq \mathcal{O}_K$.

However, it is not true that $S_{K_0}/E(u) \to K$ is injective (it is always surjective). We can verify this by using the simplest example, when $E(u) = u - p$. In this case, we have

$$S_{K_0} = \left\{ \sum_{i=0}^{\infty} a_i (u-p)^i : a_i \in K_0, v(a_i) + v(i!) \text{ is bounded below} \right\}$$

The kernel of the natural map $S_{K_0} \to K$, $u \mapsto p$ consists of those elements that have $a_0 = 0$. However, it is not true that this kernel is generated by $(u-p)$ *in* $S_{K_0}$. Indeed, we have

$$\sum_{i=1}^{\infty} a_i (u-p)^i = (u-p) \sum_{i=1}^{\infty} a_i (u-p)^{i-1}$$

and the element $\sum_{i=1}^{\infty} a_i (u-p)^{i-1}$ does *not* in general belong to $S_{K_0}$. For example, this is the case if we take $a_i$ to be zero if $i$ is not a power of $p$ and $a_{p^n} = 1/(p^n)!$ for all $n \geq 0$.

The ring $S^{pd}$, on the other hand, does not suffer from this shortcoming:

**Lemma 3.3.2.** *Let $P(u) \in W[u]$ be a monic polynomial of degree $d$. Suppose that every coefficient other than the leading one is divisible by $p$. Suppose that $d \leq e$. Then the natural map $K_0[u]/P(u)^r \to S^{pd}_{K_0}/P(u)^r$ is an isomorphism for every $r \geq 0$. In particular, the map $u \mapsto \pi$ induces an isomorphism $S^{pd}_{K_0}/E(u) \simeq K$.*

*Proof.* From the description in Lemma 3.1.6, it is enough to prove that the ideal

$$\left\{ \sum_{k=1}^{\infty} a_{jk} u^j P(u)^k : v(a_{jk}) + \left\lfloor \frac{q(dk)-1}{p-1} \right\rfloor' \geq 0 \right\}$$

is generated by $P(u)$ in $S_{K_0}$. This follows because

$$\sum_{k=1}^{\infty} a_{jk} u^j P(u)^k = P(u) \sum_{k=1}^{\infty} a_{jk} u^j P(u)^{k-1}$$

and

$$\sum_{k=1}^{\infty} a_{jk} u^j P(u)^{k-1} = \sum_{k=0}^{\infty} a_{j(k+1)} u^j P(u)^k \in S_{K_0}^{pd}$$

as is easily checked. □

**Corollary 3.3.3.** *Let* $\widehat{\mathfrak{S}}_0 = K[[u-\pi]]$. *The natural map* $S^{pd} \to \widehat{\mathfrak{S}}_0$ *induces isomorphisms*

$$S_{K_0}^{pd}/E(u)^r \xrightarrow{\simeq} \widehat{\mathfrak{S}}_0/(u-\pi)^r$$

*for all* $r \geq 0$.

*Proof.* When $r = 1$ this follows from Lemma 3.3.2. If $r > 1$, note that $E(u)\widehat{\mathfrak{S}}_0 = (u-\pi)\widehat{\mathfrak{S}}_0$ and consider the following short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_{K_0}^{pd}/E(u)^{r-1} & \xrightarrow{E(u)} & S_{K_0}^{pd}/E(u)^r & \longrightarrow & S_{K_0}^{pd}/E(u) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\ 0 & \longrightarrow & \widehat{\mathfrak{S}}_0/(u-\pi)^{r-1} & \xrightarrow{E(u)} & \widehat{\mathfrak{S}}_0/(u-\pi)^r & \longrightarrow & \widehat{\mathfrak{S}}_0/(u-\pi) & \longrightarrow & 0 \end{array}$$

The result follows by induction on $r$. □

## 3.4

**Lemma 3.4.1.** *The ring $S^{pd}$ is $p$-adically complete.*

*Proof.* Let $s_n \in S$, $n = 0, 1, \ldots$. We show that $\sum_{n=0}^{\infty} s_n p^n$ exists in $S$. Write $s_n = \sum_{i=0}^{\infty} a_{ni} u^i$. Then

$$\sum_{n=0}^{\infty} s_n p^n = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{\infty} a_{ni} u^i \right) p^n = \sum_{i=0}^{\infty} \left( \sum_{n=0}^{\infty} a_{ni} p^n \right) u^i.$$

Since $v(a_{ni})$, $n = 0, 1, \ldots$ is bounded below by $-(\lfloor i/e \rfloor - 1)/(p-1)$ and $W$ is $p$-adically complete the expression in the brackets converges. Since $v(a_{ni}p^n) \geq v(a_{ni})$, we see that the valuation of the expression in brackets satisfies the required condition for the series to be in $S$. □

## 3.5

Let $\varphi : S^{pd} \to S^{pd}$ be the ring map defined by sending $u$ to $u^p$ and equal to the canonical Frobenius on $W$. We use the same name for the analogous map on $W[u]$. Since $\varphi(E(u))$ is another Eisenstein polynomial we can define the rings $S_n^{pd} = S^{pd}(\varphi^n(E(u)))$.

**Lemma 3.5.1.** *The ring $S_n^{pd}$ is a subring of $S^{pd}$ and $\varphi(S_n^{pd}) \subset S_{n+1}^{pd}$ for every $n \geq 0$.*

*Proof.* We can assume that $n = 0$. That $S_1^{pd} \subset S^{pd}$ follows directly from the definitions. If $x = \sum_i a_i u^i \in S^{pd}$, then

$$v(\varphi(a_i)) = v(a_i) \geq - \left\lfloor \frac{\lfloor i/e \rfloor - 1}{p-1} \right\rfloor' = - \left\lfloor \frac{\lfloor ip/ep \rfloor - 1}{p-1} \right\rfloor'$$

which shows that $\varphi(x) = \sum_i \varphi(a_i) u^{ip} \in S_1^{pd}$. □

**Lemma 3.5.2.** *The map*

$$1 \otimes \varphi : W[u] \otimes_{\varphi, W[u]} S^{pd} \to S_1^{pd}, \quad 1 \otimes u \mapsto u^p$$

*is an isomorphism of* $W[u]$*-algebras. In particular* $\varphi : S^{pd} \to S_1^{pd}$ *is a finite flat morphism of degree* $p$*.*

*Proof.* To prove that $1 \otimes \varphi$ is surjective, we observe that

$$\sum_{i=0}^{\infty} a_i u^i = \sum_{k=0}^{\infty} \sum_{j=0}^{ep-1} a_{j+epk} u^{j+epk} = (1 \otimes \varphi) \left( \sum_{j=0}^{ep-1} u^j \otimes \sum_{k=0}^{\infty} \varphi^{-1}(a_{j+epk}) u^{ek} \right)$$

where we can easily check that the term on the right hand side of the tensor product belongs to $S^{pd}$. Next, suppose that

$$0 = (1 \otimes \varphi) \left( \sum_{j=0}^{p-1} u^j \otimes \sum_{k=0}^{\infty} a_{kj} u^k \right) = \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} a_{jk} u^{pk+j}$$

from where it follows that every $a_{jk} = 0$, which proves that $1 \otimes \varphi$ is injective. $\square$

# CHAPTER 4

# CONSTRUCTION OF $\mathcal{O}$-MODULES I

In Kisin's paper [13], a certain $(\varphi, \mathcal{O})$-module $\mathcal{M}(D)$ is defined for each filtered $\varphi$-module $D$. Roughly speaking, this module is obtained from the trivial vector bundle $\mathcal{O} \otimes D$ over the unit rigid-analytic disk over $K_0$ by allowing poles at the points $E(u^{p^n})$ of order $h$ on $\mathrm{Fil}^h D_K$ (see (4.1) for the precise definition).

In this chapter we present an alternative construction of $\mathcal{M}(D)$ using the ring $S^{pd}$ (and other closely related rings) suitable to be compared with an analogous, sheaf-theoretic, construction in Chapter 5.

Let us now give an overview of this construction. We will be playing with two different objects. On the one hand we will produce a $(\varphi, \mathcal{O})$-module $\mathcal{M}(D, S_\bullet^{rig})$, which will be isomorphic to $\mathcal{M}(D)$ (see definitions (4.4), (4.7), and Proposition 4.3.1). This object involves a ring $S^{rig}$ closely related to the ring $S^{pd}$ of Chapter 3. The reason for introducing this ring will be explained below. On the other hand we will also produce an object $C(D, S_\bullet^{pd})$ of the derived category $D(\mathcal{O}, \varphi)$ (see definition (4.6)) that will be $\varphi$-quasi-isomorphic to $\mathcal{M}(D, S_\bullet^{rig})$ seen as a complex concentrated in degree 0 (Corollary 4.4.5). In the next chapter we will study a similar, sheaf-theoretic, construction of this latter object in the case when $D$ arises from a crystal on a scheme $X/\mathcal{O}_K$. Putting everything together we will have succeeded in constructing $\mathcal{M}(D)$ in a *cohomological* way in terms of $X$.

Let us explain briefly the reason for introducing the ring $S^{rig}$. The complex $C(D, S_\bullet^{pd})$ is defined as a certain $R\varprojlim$ (again, see (4.6)). In order to compare it with $\mathcal{M}(D, S_\bullet^{rig})$ it is necessary to know that this (derived) inverse limit is well-behaved (concretely, that the cohomology groups of the system satisfy a Mittag-Leffler condition). This we could not prove when dealing with the ring $S^{pd}$. Hence we introduce the ring $S^{rig}$ as a rigid-analytic version of $S^{pd}$ that *does* have the right property (Corollary 4.4.4). A cofinality argument (Lemma 4.5.1) shows that replacing $S^{pd}$ by $S^{rig}$ does not change the resulting $R\varprojlim$.

## 4.1

In this section we review the definition of the $(\varphi, \mathcal{O})$-module $\mathcal{M}(D)$ attached to a filtered $\varphi$-module $D$, as given in Kisin [13].

Fix an algebraic closure $\overline{K}$ of $K$ and a sequence of elements $\pi_n \in \overline{K}$, for $n$ a non-negative integer, such that $\pi_0 = \pi$, $\pi_{n+1}^p = \pi_n$. Write $K_{n+1} = K(\pi_n)$. Let $\widehat{\mathfrak{S}}_n = K_{n+1}[[u - \pi_n]]$, $n \geq 0$. We always consider $\widehat{\mathfrak{S}}_n$ as a filtered ring using the $(u - \pi_n)$-adic filtration.

**Definition 4.1.1.** *A* filtered $\varphi$-module over $K$ *is a triple* $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ *consisting of a finite dimensional* $K_0$*-vector space* $D$*, a* $K_0$*-semi-linear bijection* $\varphi : \varphi^* D \to D$ *and a decreasing, exhaustive, and separated filtration* $\mathrm{Fil}^q D_K$ *on* $D_K = D \otimes_{K_0} K$ *by* $K$*-subspaces.*

Let $\mathcal{O}$ be the ring of rigid analytic functions on the open unit disk over $K_0$, i.e., the subring of $K_0[[u]]$ consisting of those power series $\sum_{i=0}^{\infty} a_i u^i$ such that they converge for every $|u| < 1$. Let $c_0 = E(u)$ and put

$$\lambda = \prod_{n=0}^{\infty} \varphi^n(E(u)/c_0) \in \mathcal{O}.$$

Let $D$ be a filtered $\varphi$-module. We define a $\varphi$-module over $\mathcal{O}$ as follows: for each non-negative integer $n$, write $\iota_n$ for the composite

$$\mathcal{O} \otimes_{K_0} D \xrightarrow{\varphi_W^{-n} \otimes \varphi^{-n}} \mathcal{O} \otimes_{K_0} D \longrightarrow \widehat{\mathfrak{S}}_n \otimes_{K_0} D = \widehat{\mathfrak{S}}_n \otimes_K D_K,$$

where the second map is deduced from the map $\mathcal{O} \to \widehat{\mathfrak{S}}_n$ given by taking the Taylor expansion at $u = \pi_n$. We may extend this to a map

$$\iota_n : \mathcal{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n\left[\frac{1}{u-\pi_n}\right] \otimes_K D_K.$$

Set

$$\mathcal{M}(D) := \left\{x \in \mathcal{O}\left[\frac{1}{\lambda}\right] \otimes_{K_0} D : \iota_n(x) \in \mathrm{Fil}^0\left(\widehat{\mathfrak{S}}_n\left[\frac{1}{u-\pi_n}\right] \otimes_K D_K\right), n \geq 0\right\}. \quad (4.1)$$

This is a finite free $\mathcal{O}$-module [13, Lemma 1.2.2] and comes equipped with a semi-linear map induced by $\varphi \otimes \varphi : \mathcal{O} \otimes D \to \mathcal{O} \otimes D$. Hence we have a functor:

$$\mathcal{M} : \{\text{filtered } \varphi\text{-modules}\} \to \{(\varphi, \mathcal{O})\text{-modules of finite } E\text{-height}\}.$$

## 4.2

In this section we study a construction of $(\varphi, \mathcal{O})$-modules attached to a filtered $\varphi$-modules $D$ and certain sequences of subrings of $K_0[[u]]$ (an "admissible system", see Definition 4.2.1). At the same time, we also define some complexes that, for some well-chosen admissible systems, represent $\mathcal{M}(D)$ as above. We will be mainly concerned with two different admissible systems, one constructed with the ring $S^{pd}$ of Chapter 3, and another constructed with affinoid algebras $S_n^{rig}$ corresponding to a sequence of closed disks inside $\mathcal{O}$ with radii tending to one. The complexes corresponding to the system built out of the ring $S^{pd}$ will be the output of the cohomological construction that will be studied in Chapter 5. As explained in the introduction to this chapter, we cannot show directly that the cohomology of this complex computes $\mathcal{M}(D)$. We can, however, show that the cohomology of a similar complex built using $S_n^{rig}$ *does* compute $\mathcal{M}(D)$ (Corollary 4.4.5). Finally, a cofinality argument (Lemma 4.5.1) shows that both complexes are in fact isomorphic.

**Definition 4.2.1.** *An* admissible system *will be a sequence of $\mathcal{O}$-subalgebras*

$$\cdots \subseteq S_{n+1} \subseteq S_n \subseteq \cdots \subseteq S_0 \subseteq K_0[[u]]$$

*such that $\mathcal{O} = \bigcap S_n$ and the usual $K_0$-semi-linear $\varphi : K_0[[u]] \to K_0[[u]]$ restricts to a* flat *map*

$$\varphi : S_n \to S_{n+1}$$

*for every $n \geq 0$. We will also assume that $\varphi^{n+1}(E(u))$ is a unit in $S_n$ and that the map $\mathcal{O} \to \widehat{\mathfrak{S}}_n$ extends to $S_n$ (the extension is necessarily unique).*

*Remark* 4.2.2. An example of an admissible system is given by

$$S_n^{pd} = S^{pd}(\varphi^n(E(u)))_{K_0} = K_0[u] \otimes_{\varphi^n, K_0[u]} S^{pd}$$

(see Corollary 3.3.3 and Lemma 3.5.1).

We will denote such a system simply by $S_\bullet$. Given a filtered $\varphi$-module $D$ and $n \geq 0$, define a $S_n$-module by

$$M_n(D, S_\bullet) := S_n \otimes \varphi^{n*} D.$$

There is a natural map

$$\iota_n : M_n(D, S_\bullet) = S_n \otimes_{\varphi^n} D \xrightarrow{\varphi_W^{-n} \otimes 1} \widehat{\mathfrak{S}}_n \otimes_{K_0} D = \widehat{\mathfrak{S}}_n \otimes_K D_K. \tag{4.2}$$

Since $\pi_n$ is a root of $\varphi_W^{-n} \varphi^n(E(u))$ we may extend this to a map

$$\iota_n : M_n(D, S_\bullet)\left[\frac{1}{\varphi^n(E(u))}\right] \longrightarrow \widehat{\mathfrak{S}}_n\left[\frac{1}{u - \pi_n}\right] \otimes_K D_K. \tag{4.3}$$

We define an $S_n$-module $D(S,n)^+$ by the formula

$$M_n(D,S_\bullet)^+ := \left\{x \in M_n(D,S_\bullet)\left[\frac{1}{\varphi^n(E(u))}\right] : \iota_n(x) \in \mathrm{Fil}^0\left(\widehat{\mathfrak{S}}_n\left[\frac{1}{u-\pi_n}\right] \otimes_K D_K\right)\right\}. \quad (4.4)$$

*Remark* 4.2.3. If $h \geq 0$ is chosen so that $\mathrm{Fil}^h D_K = 0$, then it follows from the definition that

$$M_n(D,S_\bullet)^+ \subseteq \lambda^{-h} S_n \otimes \varphi^{n*} D.$$

There is a natural injective $S_{n+1}$-linear map

$$1 \otimes \varphi : M_{n+1}(D,S_\bullet) \to M_n(D,S_\bullet)$$

Since $\varphi^{n+1}(E(u))$ is a unit in $S_n$, we may extend this map to an injective map

$$1 \otimes \varphi : M_{n+1}(D,S_\bullet)^+ \to M_n(D,S_\bullet).$$

Let $\lambda_{-1} = 1$ and $\lambda_n = E(u) \cdots \varphi^n(E(u))$ for $n \geq 0$. Let $A_n(D,S_\bullet)$, $B_n(D,S_\bullet)$ be defined by

$$A_n(D,S_\bullet)^+ = \bigoplus_{i=0}^{n} M_i(D,S_\bullet)^+\left[\frac{1}{\lambda_{i-1}}\right], \qquad B_n(D,S_\bullet) = \bigoplus_{i=0}^{n-1} M_i(D,S_\bullet)\left[\frac{1}{\lambda_i}\right] \quad (4.5)$$

The map $1 \otimes \varphi : M_{n+1}(D,S_\bullet)^+ \to M_n(D,S_\bullet)$ together with the inclusion $M_i(D,S_\bullet)^+ \subset M_i(D,S_\bullet)[1/\varphi^i(E(u))]$ gives a map

$$h_n : A_n(D,S_\bullet) \to B_n(D,S_\bullet)$$
$$(x_i)_{0\leq i\leq n} \longmapsto ((1\otimes\varphi)(x_1) - x_0, \ldots, (1\otimes\varphi)(x_{n-1}) - x_{n-2}, (1\otimes\varphi)(x_n) - x_{n-1}).$$

In what follows, we will write this formula simply as $h_n(x) = (1 \otimes \varphi)(x) - x$. Define

$$J_n(D, S_\bullet) := \ker h_n \subseteq M_n(D, S_\bullet)\left[\frac{1}{\lambda_n}\right] \quad \text{and} \quad C_n(D, S_\bullet) := \operatorname{Cone} h_n[-1].$$

(note that $C_n(D, S_\bullet)$ is concentrated in degrees 0 and 1). The projections on the first $n-1$ components of the direct sums (4.5) induce maps

$$J_{n+1}(D, S_\bullet) \to J_n(D, S_\bullet) \quad \text{and} \quad C_{n+1}(D, S_\bullet) \to C_n(D, S_\bullet).$$

The map $\varphi : S_n \to S_{n+1}$ induces a semi-linear map

$$\varphi \otimes 1 : M_n(D, S_\bullet) \to M_{n+1}(D, S_\bullet).$$

Since $\varphi(\lambda_n)E(u) = \lambda_{n+1}$, it extends to a map

$$\varphi \otimes 1 : M_n(D, S_\bullet)^+ \to M_{n+1}(D, S_\bullet)^+.$$

It is easy to check that the induced map on $A_n(D, S_\bullet)$ and $B_n(D, S_\bullet)$ commutes with $h_n$ and therefore we obtain a (semi-linear) map

$$\varphi \otimes 1 : J_n(D, S_\bullet) \to J_{n+1}(D, S_\bullet) \quad \text{and} \quad \varphi \otimes 1 : C_n(D, S_\bullet) \to C_{n+1}(D, S_\bullet).$$

Finally, we put

$$\mathcal{M}(D, S_\bullet) := \varprojlim J_n(D, S_\bullet) \quad \text{and} \quad C(D, S_\bullet) := R\varprojlim C_n(D, S_\bullet). \tag{4.6}$$

Then $M(D, S_\bullet)$ is naturally a $(\varphi, \mathcal{O})$-module and $C(D, S_\bullet)$ is an object of the derived category $D(\mathcal{O}, \varphi)$.

## 4.3

In this section we introduce a certain "rigid-analytic" admissible system and prove that its associated $(\varphi, \mathcal{O})$-module computes $\mathcal{M}(D)$. Define a sequence of rings by

$$S_n^{\text{rig}} := \left\{ \sum_{i=0}^{\infty} a_i u^i \in K_0[[u]] : \text{converges in } |u| \leq p^{-1/ep^n(p-1)} \right\}. \tag{4.7}$$

Clearly $\mathcal{O} \subseteq S_{n+1}^{rig} \subseteq S_n^{rig} \subseteq K_0[[u]]$. If $\sum_{i=0}^{\infty} a_i u^i \in S_n^{rig}$, we have

$$v(\varphi(a_i)u^{pi}) = v(a_i) + piv(u)$$

from where it follows that $\varphi(S_n^{rig}) \subseteq S_{n+1}^{rig}$.

**Proposition 4.3.1.** *The maps $1 \otimes \varphi^{-n} : \mathcal{O} \otimes D \to S_n^{rig} \otimes \varphi^{n*}D$ induce an isomorphism*

$$\alpha : \mathcal{M}(D) \xrightarrow{\simeq} \mathcal{M}(D, S_\bullet^{rig}).$$

*Proof.* By inspection of the definitions, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}\left[\frac{1}{\lambda}\right] \otimes D & \xrightarrow{1\otimes\varphi^{-n}} & S_n^{rig}\left[\frac{1}{\varphi^n(E(u))}\right] \otimes \varphi^{n*}D \\ & \searrow^{\iota_n} \quad \swarrow^{\iota_n^{rig}} & \\ & \widehat{\mathfrak{S}}_n\left[\frac{1}{u-\pi_n}\right] \otimes_K D_K. & \end{array} \tag{4.8}$$

It follows that $1 \otimes \varphi^{-n}$ induces a map

$$\mathcal{M}(D) \to M_n(D, S_\bullet^{rig})^+ \left[\frac{1}{\lambda_{n-1}}\right]$$

for every $n$. These maps are clearly injective. It follows from Remark 4.2.3 that

$$J_n^{rig} \subseteq \lambda^{-h} S_n^{rig} \otimes \varphi^{n*} D.$$

Thus

$$\mathcal{M}(D, S_\bullet^{rig}) \subseteq \varprojlim \lambda^{-h} S_n^{rig} \otimes \varphi^{n*} D \underset{\alpha}{\overset{\simeq}{\longleftarrow}} \lambda^{-h} \mathcal{O} \otimes D \subseteq \mathcal{O}\left[\frac{1}{\lambda}\right] \otimes D.$$

Let $x \in \mathcal{M}(D, S_\bullet^{rig})$. Then from the previous considerations we see that there exists $y \in \mathcal{O}[1/\lambda] \otimes D$ such that $\alpha(y) = x$. The commutative diagram (4.8) shows that $y \in \mathcal{M}(D)$. □

## 4.4

In this section we will see that the (derived) inverse limit of the cones $C_n(D, S_\bullet^{rig})$ associated to $D$ and the rigid-analytic admissible system introduced in the previous section are concentrated in degree 0 and compute $\mathcal{M}(D)$ in this degree.

*Remark* 4.4.1. If $P(u) \in K_0[u]$ is an irreducible polynomial with a root of absolute value $\leq p^{-1/ep^n(p-1)}$, then $S_n^{rig}/P(u) \simeq K_0[u]/P(u)$ (this follows from considering MaxSpec $S_n^{rig}$ as a rigid analytic disk). If $Q(u) \in K_0[u]$ is another such polynomial, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0[u]/P(u) & \xrightarrow{Q(u)} & K_0[u]/P(u)Q(u) & \longrightarrow & K_0[u]/Q(u) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_n^{rig}/P(u) & \xrightarrow{Q(u)} & S_n^{rig}/P(u)Q(u) & \longrightarrow & S_n^{rig}/Q(u) & \longrightarrow & 0 \end{array}$$

shows that the same is true for $P(u)Q(u)$.

**Lemma 4.4.2.** *The maps $h_n : A_n(D, S_\bullet^{rig})^+ \to B_n(D, S_\bullet^{rig})$ are surjections for every $n \geq 1$. In particular there are quasi-isomorphisms of $(\varphi, \mathcal{O})$-modules*

$$J_n(D, S_\bullet^{rig}) \underset{qis}{\xrightarrow{\simeq}} C_n(D, S_\bullet^{rig})$$

*for each $n \geq 1$.*

*Proof.* We prove this by induction on $n$. Suppose first that $n = 1$. Since $M_n(D, S_\bullet^{rig}) \subseteq M_n(D, S_\bullet^{rig})^+$ for every $n \geq 0$, it is enough to prove that the map

$$h_1 : M_1(D, S_\bullet^{rig}) \left[\frac{1}{E(u)}\right] \oplus M_0(D, S_\bullet^{rig}) \to M_0(D, S_\bullet^{rig}) \left[\frac{1}{E(u)}\right]$$

given by $(x, y) \mapsto (1 \otimes \varphi)(x) - y$ is surjective. Let $z_0 \in M_0(D, S_\bullet^{rig})[1/E(u)]$ and let $r \geq 0$ be such that $E(u)^r z_0 \in M_0(D, S_\bullet^{rig}) = S_0^{rig} \otimes D$. It follows from Remark 4.4.1 that the map

$$1 \otimes \varphi : M_1(D, S_\bullet^{rig})/E(u)^r \to M_0(D, S_\bullet^{rig})/E(u)^r$$

is an isomorphism (note that the root $u = \pi$ of $E(u)$ lies inside all the $S_n^{rig}$). Thus, there is $w_1' \in M_1(D, S_\bullet^{rig})$, $w_0 \in M_0(D, S_\bullet^{rig})$ such that

$$(1 \otimes \varphi)(w_1') = E(u)^r z_0 + E(u)^r w_0.$$

Then $h_1(w_1' E(u)^{-r}, w_0) = z_0$ and we are done with this case.

Suppose now that $n > 1$. Again it is enough to prove that the map

$$h_n : \bigoplus_{i=0}^{n} M_i(D, S_\bullet^{rig}) \left[\frac{1}{\lambda_{i-1}}\right] \to \bigoplus_{i=0}^{n-1} M_i(D, S_\bullet^{rig}) \left[\frac{1}{\lambda_i}\right]$$

is surjective. Let $(z_0, \ldots, z_{n-1}) \in \bigoplus_{i=0}^{n-1} M_i(D, S_\bullet)[1/\lambda_i]$. By induction we know there are $(w_0', \ldots, w_{n-1}') \in \bigoplus_{i=0}^{n-1} M_i(D, S_\bullet^{rig})[1/\lambda_{i-1}]$ such that

$$h_{n-1}(w_0', \ldots, w_{n-1}') = (z_0, \ldots, z_{n-2}).$$

*Claim.* There are $w_n \in M_n(D, S_\bullet^{rig})[1/\lambda_{n-1}]$ and $t \in M_{n-1}(D, S_\bullet^{rig})$ such that

$$(1 \otimes \varphi)(w_n) - (w'_{n-1} + t) = z_{n-1}.$$

Let $r \geq 0$ be such that $\lambda_{n-1}^r z_{n-1} \in M_{n-1}(D, S_\bullet^{rig})$ and $\lambda_{n-1}^r w'_{n-1} \in M_{n-1}(D, S_\bullet^{rig})$. As before, we have an isomorphism

$$1 \otimes \varphi : M_n(D, S_\bullet^{rig})/\lambda_{n-1}^r \xrightarrow{\simeq} M_{n-1}(D, S_\bullet^{rig})/\lambda_{n-1}^r.$$

We deduce that there exists $w'_n \in M_n(D, S_\bullet^{rig})$ and $t \in M_{n-1}(D, S_\bullet^{rig})$ such that

$$(1 \otimes \varphi)(w'_n) - \lambda_{n-1}^r t = \lambda_{n-1}^r (z_{n-1} + w'_{n-1}).$$

The claim follows putting $w_n = \lambda_{n-1}^{-r} w'_n$.

Finally, let $w_i = w'_i + (1 \otimes \varphi)^{n-i-1} t$, $0 \leq i < n$. It is easy to verify that $h_n(w_0, \ldots, w_n) = (z_0, \ldots, z_{n-1})$. □

**Lemma 4.4.3.** *The module $J_n(D, S_\bullet^{rig})$ is finite free over $S_n^{rig}$ and the natural map $J_{n+1}(D, S_\bullet^{rig}) \otimes S_n^{rig} \to J_n(D, S_\bullet^{rig})$ is an isomorphism.*

*Proof.* We think of the modules $M_n(D, S_\bullet^{rig})^+$ as sheaves on the affinoid MaxSpec $S_n^{rig}$. Since $M_n(D, S_\bullet^{rig})^+ \subset \lambda^{-h} S_n^{rig} \otimes \varphi^{n*} D$ (Remark 4.2.3) it follows that $M_n(D, S_\bullet^{rig})$ are finite free $S_n^{rig}$-modules. We can think of $J_n(D, S_\bullet^{rig})$ as the module of global sections of the sheaf obtained by pasting together these sheaves. Hence it is finitely generated and torsion free, hence finite free. The second statement also follows similarly from these geometric considerations. □

**Corollary 4.4.4.** *The system $n \mapsto J_n(D, S_\bullet^{rig})$ is acyclic for $R \varprojlim$.*

*Proof.* This follows from Lemma 4.4.3 by a Mittag-Leffler argument. See for example [17, p. 9, Theorem]. □

**Corollary 4.4.5.** *With notation as above, there is a quasi-isomorphism of* $(\varphi, \mathcal{O})$*-modules*

$$R\varprojlim C_n(D, S_\bullet^{rig}) \xrightarrow[qis]{\simeq} \varprojlim J_n(D, S_\bullet^{rig}) \simeq \mathcal{M}(D, S_\bullet^{rig}).$$

*Proof.* This follows from Lemma 4.4.2 and Corollary 4.4.4. □

## 4.5

Finally, in this section we compare $C(D, S_\bullet^{rig}) = R\varprojlim C_n(D, S_\bullet^{rig})$ (which we know computes $\mathcal{M}(D)$) and $C(D, S_\bullet^{pd}) = R\varprojlim C_n(D, S_\bullet^{pd})$ (which will be the output of the cohomological construction in Chapter 5) and see that they are quasi-isomorphic.

An admissible system $S_\bullet$ *is cofinal in* $T_\bullet$ if $S_{n+1} \subseteq T_n \subseteq S_n$ for every $n \geq 0$. For example, $S_\bullet^{rig}$ is cofinal in $S_\bullet^{pd}$.

**Lemma 4.5.1.** *Let* $S_\bullet$ *be cofinal in* $T_\bullet$*. Then the natural map* $C_n(D, T_\bullet) \to C_n(D, S_\bullet)$ *induces an isomorphism*

$$R\varprojlim C_n(D, T_\bullet) \xrightarrow{\simeq} R\varprojlim C_n(D, S_\bullet)$$

*in* $D(\mathcal{O}, \varphi)$.

*Proof.* The inclusion $S_{n+1} \subset T_n$ gives a map

$$1 \otimes \varphi : M_{n+1}(D, S_\bullet) \to M_n(D, T_\bullet)$$

and it is easy to see that it extends to a map

$$1 \otimes \varphi : C_{n+1}(D, S_\bullet) \to C_n(D, T_\bullet).$$

The composition of $1 \otimes \varphi$ with the map $C_n(D, T_\bullet) \to C_n(D, S_\bullet)$ induced by the inclusion $T_n \subset S_n$ is precisely the transition homomorphism $C_{n+1}(D, S_\bullet) \to C_n(D, S_\bullet)$. Similarly, the composition of the map $C_n(D, T_\bullet) \to C_n(D, S_\bullet)$ with $1 \otimes \varphi$ gives the transition homomorphism $C_n(D, T_\bullet) \to C_{n-1}(D, T_\bullet)$. Since for any inverse system $L$ the natural map $L_{n+1} \to L_n$ induces an isomorphism $R \varprojlim_n L_{n+1} \simeq R \varprojlim_n L_n$, the result follows. □

**Corollary 4.5.2.** *There is a quasi-isomorphism of* $(\varphi, \mathcal{O})$*-modules*

$$\mathcal{M}(D) \xrightarrow[\simeq]{qis} C(D, S_\bullet^{pd}).$$

*Proof.* This follows from Proposition 4.3.1, Corollary 4.4.5, and Lemma 4.5.1. □

# CHAPTER 5

# CONSTRUCTION OF $\mathcal{O}$-MODULES II

Let $X$ be a proper and smooth scheme over $\mathcal{O}_K$. In this chapter we "lift" the construction of Chapter 4 to the level of sheaves on $X$. The objective is to construct some complexes of $(\varphi, \mathcal{O})$-modules on $X$ (or rather its special fiber) whose cohomology computes the $(\varphi, \mathcal{O})$-module $\mathcal{M}(D)$ associated to a filtered $\varphi$-module $D$ arising from a filtered $\varphi$-crystal on $X$ (Proposition 5.5.1).

Let us briefly describe the contents of each section. In Section 5.1 we define the analog, for sheaves, of the construction (4.4) in the case of modules. In Section 5.2, we define the sheaf-theoretic analog of (4.6).

In Section 5.3, we study the crystalline cohomology of $X$ over the base $S^{pd}$ and its Frobenius action. We prove two crucial results. Proposition 5.3.1 shows that the crystalline cohomology of $X/S^{pd}$ can be represented by a complex with zero differentials, from which we deduce that the individual cohomology groups are finite projective $S^{pd}$-modules equipped with an étale Frobenius. The same result is proven by Faltings [8] for the "usual" ring $S$. Using this result and a suitable version of Dwork's trick (Lemma 5.3.2), Corollary 5.3.3 compares $H^m_{\text{cris}}(X/S^{pd}_{K_0}, \mathcal{E})$ and $S^{pd}_{K_0} \otimes H^m_{\text{cris}}(X_0/W, \mathcal{E}_0)$, *as* $(\varphi, S^{pd})$*-modules.*

In Section 5.4 we study the PD-filtration on the crystalline cohomology of $X/S^{pd}$. We show (Propositions 5.4.2 and 5.4.3) that the PD-filtration on $H^m_{\text{cris}}(X/S^{pd}, \mathcal{E})$ is obtained (after inverting $p$) by pulling back the Hodge filtration of $H^m_{dR}(X_K/K, \mathcal{E}_K)$ via a map similar to $\iota_0$ of (4.2). Together with the results of Section 5.3 these results imply that the construction presented in Sections 5.1 and 5.2 produces, after taking (derived) global sections, the object denoted by $C(D, S_\bullet^{pd})$ (where $D = H^m_{\mathrm{cris}}(X_0/W, \mathcal{E}_0)[1/p]$) in Section 4.2. This object was proven to be $\varphi$-quasi-isomorphic to $\mathcal{M}(D)$.

Finally, in Section 5.5, we put everything together. The main result (Proposition 5.5.1) follows easily from the previous results of this chapter and those of Chapter 4.

## 5.1

Let $S$ be a filtered ring (as always, the filtration satisfies $\operatorname{Fil}^0 S = S$). Let $K$ be a filtered $S$-complex. Let $\lambda \in \operatorname{Fil}^1 S$. We consider $\lambda$ as a morphism $\lambda : \operatorname{Fil}^n K \to \operatorname{Fil}^{n+1} K$ and we define

$$\begin{aligned} K(\lambda) &= \operatorname{coker}\left(1 - \lambda : \bigoplus_{n=0}^{\infty} \operatorname{Fil}^n K \to \bigoplus_{n=0}^{\infty} \operatorname{Fil}^n K\right) \\ &= \varinjlim \left(\operatorname{Fil}^0 K \xrightarrow{\lambda} \operatorname{Fil}^1 K \xrightarrow{\lambda} \cdots\right). \end{aligned}$$

The maps $\operatorname{Fil}^n K \to K[1/\lambda]$, $x \mapsto \lambda^{-n} x$ induce an isomorphism

$$K(\lambda) \simeq \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \operatorname{Fil}^n K \subseteq K\left[\frac{1}{\lambda}\right].$$

The short exact sequence

$$0 \to \bigoplus_{n=0}^{\infty} \operatorname{Fil}^n K \xrightarrow{1-\lambda} \bigoplus_{n=0}^{\infty} \operatorname{Fil}^n K \to K(\lambda) \to 0 \tag{5.1}$$

induces a long exact sequence of cohomology groups. Since $1 - \lambda$ remains injective after taking cohomology, the resulting sequence splits in short exact sequences and thus it follows that

$$H^q(K(\lambda)) = \sum_{m=0}^{\infty} \lambda^{-m} \operatorname{Fil}^m H^q(K) \subseteq H^q(K)[1/\lambda],$$

where $\operatorname{Fil}^m H^q(K) = \operatorname{im}(H^q(\operatorname{Fil}^m K) \to H^q(K))$.

**Lemma 5.1.1.** *Let $S$, $T$ be filtered rings (considered as constant sheaves). Let $f : (X, S) \to (Y, T)$ is a morphism of ringed spaces. Let $\lambda \in \mathrm{Fil}^1 T$. Then there is a commutative diagram*

$$\begin{array}{ccc} D^+F(X,S) & \xrightarrow{Rf_*} & D^+F(Y,T) \\ {\scriptstyle (-)(\lambda)}\downarrow & & \downarrow{\scriptstyle (-)(\lambda)} \\ D^+(X,S) & \xrightarrow{Rf_*} & D^+(Y,T) \end{array}$$

*Proof.* It follows from the injectivity of $1 - \lambda$ in (5.1) that we have an isomorphism of functors $f_*(-)(\lambda) \simeq f_*((-)(\lambda))$. The claim follows then from the observation that $(-)(\lambda)$ sends flasque sheaves to flasque sheaves. $\square$

## 5.2

Let $X$ be a topological space. Let $S_\bullet$ be an admissible system (cf. Section 4.2). We denote as usual $S_0$ by $S$. Given a $(\varphi, S)$-complex $K$ on $X$, we put $M_n(K, S_\bullet) = S_n \otimes_{\varphi^n, S} K$, $M_n(K, S_\bullet)^+ = S_n \otimes_{\varphi^n, S} K(E(u))$. As usual, let $\lambda_{-1} = 1$, $\lambda_i = \varphi^i(E(u))\lambda_{i-1}$ if $i \geq 0$. Next, define $A_n(K, S_\bullet)$ and $B_n(K, S_\bullet)$ by

$$A_n(K, S_\bullet) := \bigoplus_{i=0}^{n} M_i(K, S_\bullet)^+ \left[\frac{1}{\lambda_{i-1}}\right], \quad B_n(K, S_\bullet) := \bigoplus_{i=0}^{n-1} M_i(K, S_\bullet)\left[\frac{1}{\lambda_i}\right].$$

The Frobenius $\varphi : K \to K$ induces $S_n$-linear maps

$$1 \otimes \varphi : S_n \otimes_{\varphi^n} K \to S_{n-1} \otimes_{\varphi^{n-1}} K.$$

Since $\varphi(E(u)) \in S_{K_0}^\times$, these extend to maps

$$1 \otimes \varphi : M_i(K, S_\bullet)^+ \to M_{i-1}(K, S_\bullet)$$

which in turn gives a map

$$1 \otimes \varphi : A_n(K, S_\bullet) \to B_n(K, S_\bullet).$$

The inclusions $M_i(K, S_\bullet)^+ \subset M_i(K, S_\bullet)[\varphi^i(E(u))^{-1}]$ define a map $\iota : A_n(K, S_\bullet) \to B_n(K, S_\bullet)$. Let

$$C_n(K, S_\bullet) := \mathrm{Cone}\,(1 \otimes \varphi - \iota : A_n(K, S_\bullet) \to B_n(K, S_\bullet))\,[-1].$$

Let $\varphi : M_i(K, S_\bullet) \to M_{i+1}(K, S_\bullet)$ be the $\varphi$-semi-linear map

$$\varphi : M_i(K, S_\bullet) \to M_{i+1}(K, S_\bullet), \quad x \mapsto 1 \otimes x.$$

It induces maps

$$\varphi : C_n(K, S_\bullet) \to C_{n+1}(K, S_\bullet).$$

The $C_n(K, S_\bullet)$ form an inverse system in $D(X, S_\bullet, \varphi_\bullet)$ and we put

$$\mathcal{M}(K, S_\bullet) := R \varprojlim_n C_n(K, S_\bullet) \in D(X, \mathcal{O}, \varphi). \tag{5.2}$$

**Lemma 5.2.1.** *Let $S_\bullet$ be an admissible system. Let $f : (X, S_\bullet, \varphi_\bullet) \to (Y, S_\bullet, \varphi_\bullet)$ be a homomorphism. Suppose that the natural map $S_n \otimes_{\varphi^n} Rf_*(K) \to Rf_*(S_n \otimes_{\varphi^n} K)$ is an isomorphism for every $n \geq 0$ (recall that $\varphi^n : S_0 \to S_n$ is flat for every $n \geq 0$). Then there is a canonical isomorphism*

$$\mathcal{M}(Rf_*(K), S_\bullet) \xrightarrow{\simeq} Rf_*(\mathcal{M}(K, S_\bullet))$$

*of $(\varphi, \mathcal{O})$-modules.*

*Proof.* This follows directly from the definitions and the tautological isomorphism $(S_n \otimes_{\varphi^n} K)(\varphi^n(E(u))) \simeq S_n \otimes_{\varphi^n} K(E(u))$. □

## 5.3

We next study the Frobenius action on the crystalline cohomology groups $H^*_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})$. Refer to the introduction to this chapter for more details.

**Proposition 5.3.1.** *Let $\mathcal{E}$ be a filtered $\varphi$-crystal on $X/S^{pd}$. Suppose that the Hodge-de Rham spectral sequence degenerates for $\mathcal{E}_X[1/p]$ (a flat vector bundle on $X_K$). Then the filtered perfect complex $R\Gamma_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0}$ can be represented by a complex with zero differentials in $D^+F(X, S^{pd})$.*

*Proof.* This is contained in [8, p. 120]. There it is shown that $R\Gamma_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})$ can be represented by a bounded complex $L$ of finite filtered $(\varphi, S)$-modules such that the differentials at $u = \pi$ of $L[1/p]$ are zero (because of the degeneration condition). On the other hand, Frobenius induces a quasi-isomorphism

$$\varphi : S^{pd}[1/p] \otimes_{\varphi, S^{pd}} L \xrightarrow{\simeq} L$$

(this follows, for example, from [20, Proposition 2.24]). The assumption that $L$ has zero differentials at $u = \pi$ means that $\dim_K H^*(L/E(u)L)[1/p] = \dim_K (L/E(u)L)[1/p]$. We deduce that $\dim_K H^*(\varphi^* L/E(u)\varphi^* L)[1/p] = \dim_K \varphi^* L/E(u)\varphi^* L[1/p]$. Therefore the differentials of $\varphi^* L$ are zero at $u = \pi$. That is, the differentials of $L$ also vanish at $u = \pi^p$. Iterating this argument we see that the differentials of $L$ are zero at $\pi^{p^n}$ for $n = 0, 1, 2, \ldots$. Since the differentials are given by matrices with entries in the ring of functions of a rigid analytic disk (of radius $\leq p^{-1/e(p-1)}$), we see that they must be identically zero. □

**Lemma 5.3.2** (Dwork's trick)**.** *Let $M$ be a finite projective $(\varphi, S^{pd})$-module such that the map $\varphi : S^{pd}_{K_0} \otimes_\varphi M \xrightarrow{\simeq} M_{K_0}$ is an isomorphism of $S^{pd}_{K_0}$-modules. Then there is a unique isomorphism*

$$M_{K_0} \xrightarrow{\simeq} S^{pd}_{K_0} \otimes M/uM$$

*of $(\varphi, S^{pd}_{K_0})$-modules (where the $\varphi$ on the right is $\varphi \otimes \varphi$) lifting the identity modulo $u$.*

*Proof.* Begin by choosing a map $s_0 : M_{K_0}/u \to M_{K_0}$ lifting the identity mod $u$. Define $s : M_{K_0}/u \to M_{K_0}$ by

$$s = s_0 + \sum_{i=0}^{\infty} (\varphi^i \circ s_0 \circ \varphi^{-i} - \varphi^{i-1} \circ s_0 \circ \varphi^{1-i}).$$

To see that the series converges, choose a $W$-lattice $L \subset M_{K_0}/u$. Then $\varphi^{-1}(L) \subset p^{-j}L$ for some non-negative integer $j$. Since $\varphi \circ s_0 \circ \varphi^{-1} - s_0 \in uM_{K_0}$, we have $L_1 = u^{-1}(\varphi \circ s_0 \circ \varphi^{-1} - s_0)(L) \subset M_{K_0}$ so that

$$(\varphi^{i+1} \circ s_0 \circ \varphi^{-i-1} - \varphi^i \circ s_0 \circ \varphi^{-i})(L) \subseteq p^{-ij} u^{p^i} \varphi^i(L_1).$$

Since $p^{-ij}u^{p^i} \to 0$ in $S^{pd}_{K_0}$, it follows that the series converges. It is easy to see that $\varphi \circ s = s \circ \varphi$.

Suppose next that $s$ and $s'$ are two $\varphi$-equivariant sections. Then $(s - s')(M_{K_0}/u) \subset uM$ and since $(s - s') \circ \varphi^j = \varphi^j \circ (s - s')$ for $j \geq 0$ it follows that $(s - s')(M_{K_0}/u) \subset u^{p^j} M$ for every $j \geq 0$, i.e., that $s = s'$.

Write $s : S^{pd}_{K_0} \otimes M_{K_0}/uM_{K_0} \to M_{K_0}$ be the $S^{pd}_{K_0}$-linear extension of $s$. It is an isomorphism. Indeed, both the source and the target are $S_{K_0}$-free of the same rank. Since $s$ is an isomorphism at $u = 0$, it is an isomorphism in some small neighborhood of $u$. Since the $\varphi$ at both the source and the target is étale, it follows that $s$ is an isomorphism everywhere. □

**Corollary 5.3.3.** *There is a natural $S^{pd}_{K_0}$-linear isomorphism of $\varphi$-modules*

$$H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0} \simeq S^{pd}_{K_0} \otimes_{K_0} H^m_{\mathrm{cris}}(X_0/W, \mathcal{E}_0)_{K_0}.$$

*Proof.* Since $H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0}$ is finite projective over $S_{K_0}$ (Proposition 5.3.1), the proposition follows from Lemma 5.3.2 and the base change formula in crystalline cohomology. □

## 5.4

In this section we study the PD-filtration on the cohomology groups $H^*_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})$. Please refer to the introduction to this chapter for more details.

Let $\pi$ be the fixed uniformizer of $\mathcal{O}_K$.

**Lemma 5.4.1.** *Define a ring by*

$$S^{pd}_\pi = \left\{ \sum_{k=0}^{\infty} a_k u^k : a_k \in \mathcal{O}_K, v(a_k) + \left\lfloor \frac{q(k)-1}{p-1} \right\rfloor' \geq 0 \right\}.$$

*There is a natural map $S^{pd} \to S^{pd}_\pi$ sending $u$ to $u$. Moreover, there is a map $S^{pd}_\pi \to \mathcal{O}_K$ sending $u$ to $\pi$ such that the kernel has divided powers and the map $S^{pd} \to S^{pd}_\pi$ is a PD-map.*

*Proof.* The calculations are very similar to those of Chapter 3, but we carry them out anyway. It is easy to check that the above set is in fact a subring of $K_0[[u]]$. Let $\sum_k a_k u^k \in S^{pd}$. The existence of the map $S^{pd} \to S^{pd}_\pi$ sending $u$ to $u$ is immediate from the definitions.

Next, observe that if $\sum_k a_k u^k \in S^{pd}_\pi$,

$$\begin{aligned} \sum_{k=0}^{\infty} a_k u^k &= \sum_{k=0}^{\infty} a_k (u - \pi + \pi)^k = \sum_{k=0}^{\infty} a_k \sum_{h=0}^{k} \binom{k}{h} (u-\pi)^h \pi^{k-h} \\ &= \sum_{h=0}^{\infty} \left( \sum_{k=h}^{\infty} \binom{k}{h} a_k \pi^{k-h} \right) (u-\pi)^h. \end{aligned}$$

and the coefficient in the brackets has valuation $\geq v(a_k) \geq \lfloor (q(k)-1)/(p-1) \rfloor'$. In fact, running the computation backwards we see that the ring $S_\pi^{pd}$ can be written

$$S_\pi^{pd} = \left\{ \sum_{k=0}^{\infty} a_k (u-\pi)^k : a_k \in K, v(a_k) + \left\lfloor \frac{q(k)-1}{p-1} \right\rfloor' \geq 0 \right\}.$$

The existence of the map $S_\pi^{pd} \to \mathcal{O}_K$ is now clear and we see that the kernel of this map consists of those elements that can be written as above with $a_k = 0$. Let $k > 0$ and let $a_k(u-\pi)^k \in S_\pi^{pd}$. The same calculation as in Corollary 3.2.1 shows that $(a_k(u-\pi)^k)^j/j! \in S_\pi^{pd}$. Hence the kernel of the map $S_\pi^{pd} \to \mathcal{O}_K$ has divided powers and we are done. $\square$

**Proposition 5.4.2.** *Let $X/\mathcal{O}_K$ be proper and smooth and let $\mathcal{E}$ be a filtered crystal on $X/S^{pd}$ such that the Hodge-de Rham spectral sequence for $\mathcal{E}_X[1/p]$ degenerates. Then there is a natural isomorphism of filtered $\widehat{\mathfrak{S}}_0$-modules*

$$\widehat{\mathfrak{S}}_0 \otimes_{S^{pd}} H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E}) \simeq \widehat{\mathfrak{S}}_0 \otimes_K H^m_{dR}(X_K/K, \mathcal{E}_K).$$

*Proof.* Let $S_\pi^{pd}$ be the ring defined in the previous Lemma and consider the inclusion of PD-rings $S^{pd} \to S_\pi^{pd}$. By the base change theorem in crystalline cohomology, this inclusion induces an isomorphism

$$S_\pi^{pd} \otimes^L_{S^{pd}} R\Gamma_{\mathrm{cris}}(X/S^{pd}, \mathcal{E}) \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(X/S_\pi^{pd}, \mathcal{E}). \tag{5.3}$$

Note that $S_\pi^{pd}$ is a $\mathcal{O}_K$-algebra and therefore we can define a smooth lift of $X$ to $S_\pi^{pd}$ as $Y = X \otimes_{\mathcal{O}_K} S_\pi^{pd}$. Then by the fundamental property [1, V, §2, Théorème 2.3.2] of crystalline cohomology and the Künneth formula [1, V, §4, Proposition 4.1.1], we get an isomorphism that preserves the filtration

$$R\Gamma_{\mathrm{cris}}(X/S_\pi^{pd}, \mathcal{E}) \xrightarrow{\simeq} R\Gamma_{\mathrm{cris}}(S_\pi^{pd} \otimes_{\mathcal{O}_K} X/S_\pi^{pd}, \mathcal{E}) \xleftarrow{\simeq} S_\pi^{pd} \otimes^L_{\mathcal{O}_K} R\Gamma_{\mathrm{cris}}(X/\mathcal{O}_K, \mathcal{E}). \tag{5.4}$$

Putting (5.3) and (5.4) together, inverting $p$, and observing that $H^m_{\mathrm{cris}}(X/S^{pd})_{K_0}$ is flat (even free) over $S^{pd}_{K_0}$ (Corollary 5.3.3), we deduce that the map is an isomorphism and preserves the filtration. Since both sides are complete with respect to the filtration and the map induces an isomorphism on graded objects because the Hodge-de Rham spectral sequence of $\mathcal{E}_X[1/p]$ degenerates, it follows that it also induces an isomorphism on the individual steps of the filtration. $\square$

**Proposition 5.4.3.** *There is an isomorphism of filtered $(\varphi, S^{pd})$-modules*

$$H^q_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0} \simeq S^{pd}_{K_0} \otimes_W H^q_{\mathrm{cris}}(X_0/W, \mathcal{E}_0),$$

*where the filtration on the right hand side comes from the Hodge filtration using $\iota_0$ (4.3) and the filtration on the left hand side is the crystalline PD-filtration.*

*Proof.* There is a commutative diagram

$$\begin{array}{ccc} H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0} & \xrightarrow{\simeq} & S^{pd}_{K_0} \otimes H^m_{\mathrm{cris}}(X_0/W, \mathcal{E}_0) \\ \downarrow & & \downarrow \iota_0 \\ \widehat{\mathfrak{S}}_0 \otimes_{S^{pd}} H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E}) & \xrightarrow{\simeq} & \widehat{\mathfrak{S}}_0 \otimes_{K_0} H^m_{dR}(X_K/K, \mathcal{E}_K) \end{array}$$

The top map is the isomorphism of Lemma 5.3.2. The bottom map is an isomorphism of filtered $\widehat{\mathfrak{S}}_0$-modules (Proposition 5.4.2). Thus it is enough to prove that the filtration on $H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0}$ induced by the map on the left coincides with the crystalline PD-filtration. By Proposition 5.3.1, $H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0}$ is a filtered projective $S^{pd}_{K_0}$-module, and therefore we are reduced to proving the claim with $S^{pd}$ instead of $H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})$. This follows from Corollary 3.3.3. $\square$

**Corollary 5.4.4.** *Let $K = H^m_{\mathrm{cris}}(X/S^{pd}, \mathcal{E})_{K_0}$ (considered as a complex concentrated in* *degree 0) and $D = H^m_{\mathrm{cris}}(X_0/W, \mathcal{E}_0)_{K_0}$. Then the maps above induce isomorphisms*

$$M_n(K, S^{pd}_\bullet)^+ \xrightarrow{\simeq} M_n(D, S^{pd}_\bullet)^+.$$

*In particular, there are isomorphisms $\mathcal{M}(K, S^{pd}_\bullet) \xrightarrow{\simeq} C(D, S^{pd}_\bullet)$ of $(\varphi, \mathcal{O})$-modules.*

*Proof.* By Proposition 5.4.3 we have that the natural map $K \to S \otimes D$ is a filtered isomorphism (where the filtration on $S \otimes D$ is induced by $\iota_0$). Pulling back by $\varphi^n : S^{pd} \to S^{pd}_n$ we deduce that the natural map

$$M_n(K, S^{pd}_\bullet) = S_n \otimes_{\varphi^n} K \to S_n \otimes \varphi^{n*} D = M_n(D, S^{pd}_\bullet)$$

is again an isomorphism. Since $\varphi_W^{-n}\varphi^n(E(u))\widehat{\mathfrak{S}}_n = (u - \pi_n)\widehat{\mathfrak{S}}_n$ we see that

$$\sum_{k=0}^{\infty} \varphi^n(E(u))^{-k} \operatorname{Fil}^k(S^{pd}_n \otimes \varphi^{n*} D) = \operatorname{Fil}^0 \left( S^{pd}_n \left[ \frac{1}{\varphi^n(E(u))} \right] \otimes \varphi^{n*} D \right),$$

where the filtration on $S^{pd}_n \otimes \varphi^{n*} D$ is induced by $\iota_n$ (cf. (4.4)). The conclusion follows. □

## 5.5

Let $X$ be a proper, smooth $\mathcal{O}_K$-scheme. Let $(\mathcal{E}, \varphi)$ be a filtered $\varphi$-crystal on $X/S^{pd}$ such that the Hodge-de Rham spectral sequence degenerates for $\mathcal{E}_X[1/p]$. Let $D = H^m_{\mathrm{cris}}(X_0/W, \mathcal{E}_0)_{K_0}$ be the crystalline cohomology of the crystal induced on $X_0 = X \otimes k$. It comes equipped with a bijective semi-linear Frobenius $\varphi : D \to D$. Let

$$\mathcal{M}(X, \mathcal{E}) := \mathcal{M}(Ru_{\mathcal{X}/S^{pd}*}(\mathcal{E}, \varphi), S^{pd}_\bullet) \in D(X_0, \mathcal{O}, \varphi)$$

(the right hand side was defined in (5.2)). For the next proposition, the definition of $\mathcal{M}(D)$ was recalled in (4.1).

**Proposition 5.5.1.** *Let $X/\mathcal{O}_K$ be proper and smooth. Let $(\mathcal{E},\varphi)$ be a filtered $\varphi$-crystal on $X/S^{pd}$ such that the Hodge-de Rham spectral sequence for $\mathcal{E}_X[1/p]$ degenerates. Let $D = H^q(X_0/W,\mathcal{E}_0)_{K_0}$ be the filtered $\varphi$-module defined above. There is a functorial isomorphism*

$$\mathcal{M}(D) \simeq H^q(X_0, \mathcal{M}(X,\mathcal{E}))$$

*of $(\varphi,\mathcal{O})$-modules.*

*Proof.* By Lemma 5.2.1 there is a natural isomorphism

$$\mathcal{M}(R\Gamma_{cris}(\mathcal{X}/S^{pd},\mathcal{E},\varphi), S^{pd}_\bullet) \xrightarrow{\simeq} R\Gamma(X_0, \mathcal{M}(X,\mathcal{E})).$$

Thus it is enough to prove the theorem with $R\Gamma_{cris}(\mathcal{X}/S^{pd},\mathcal{E},\varphi)$ instead of $Ru_{\mathcal{X}/S^{pd}*}(\mathcal{E},\varphi)$. We start by rewriting the right hand side. Fix an isomorphism

$$R\Gamma_{\mathrm{cris}}(\mathcal{X}/S^{pd},\mathcal{E},\varphi)_{K_0} \simeq \bigoplus_q H^q_{\mathrm{cris}}(X/S^{pd},\mathcal{E})_{K_0}[-q]$$

as in Proposition 5.3.1. Then

$$\mathcal{M}(R\Gamma_{\mathrm{cris}}(\mathcal{X}/S^{pd},\mathcal{E},\varphi), S^{pd}_\bullet) \simeq \bigoplus_q \mathcal{M}(H^q_{\mathrm{cris}}(X/S^{pd},\mathcal{E})[-q], S^{pd}_\bullet).$$

It follows from Corollary 5.4.4 that there is an isomorphism

$$\mathcal{M}(H^q_{\mathrm{cris}}(X/S^{pd},\mathcal{E})[0], S^{pd}_\bullet) \simeq R\varprojlim C_n(D, S^{pd}_\bullet) =: C(D, S^{pd}_\bullet),$$

and so the result follows from Corollary 4.5.2. □

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